Anomalies and superpotential in $\mathcal{N} = 1$
noncommutative gauge theories

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The anomaly of various currents in the noncommutative supersymmetric $\mathcal{N} = 1, U(1)$
gauge theory are calculated and the effective superpotential obtained.

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1 Introduction

Noncommutative gauge theories emerged in the context of string theory in the
presence of a nontrivial background of one of the massless states of closed string,
and has been extensively studied in the last six years.

One important aspect of quantum field theories is the violation of classical
symmetries of the theory due to quantum effects, anomalies. It has been found
that in noncommutative gauge theories anomalies have essential new aspects. In
this talk the most salient features of the axial anomaly in noncommutative QED
are discussed following [1, 2].

Anomalies also play a crucial role in derivations of effective actions for super-
symmetric gauge theories.

Utilization of the axial anomalies of noncommutative gauge theories and a cor-
responding anomaly in the supersymmetric version in the derivation of an effective
action of the $\mathcal{N} = 1$ supersymmetric noncommutative $U(1)$ gauge theory are briefly
reviewed [3].

2 Anomalies

2.1 Noncommutative $U(1)$ gauge theory

Noncommutativity from string theory in the presence of background antisymmetric
field $B_{\mu\nu}$ on a brane gives (only $B_{12} \neq 0$ is assumed here)

$$[x^1, x^2] = i\Theta, \quad \text{where } \Theta \text{ is related to } B.$$  (1)

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Then the low energy of the string theory gives a noncommutative gauge theory on
the brane, where products are substituted by \(*\)-products defined by
\[
f(x) * g(x) \equiv f(x) \exp \left( \frac{i\Theta_{\mu\nu}}{2} \partial_{\mu} \partial_{\nu} \right) g(x),
\]
with the properties
\[
\exp(ikx) * \exp(ipx) = \exp\left(i(k + p)x\right) \exp\left(-\frac{1}{2} \Theta_{\mu\nu} k^\mu p^\nu\right),
\]
\[
\int f * g = \int f g,
\]
\[
\int f * g * h = \int h * f * g.
\]
The noncommutative \(U(1)\) gauge theory is given by the Lagrangian
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} * F^{\mu\nu} + \bar{\psi} * (iD - m) * \psi,
\]
with
\[
D_{\mu}\psi = \partial_{\mu}\psi + igA_{\mu} * \psi,
\]
\[
F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig(A_{\mu} * A_{\nu} - A_{\nu} * A_{\mu}).
\]
A most important property of noncommutative field theories is the UV/IR mixing,
where the better UV behavior due to the phase
\[
\exp \left( \frac{i\Theta_{\mu\nu}}{2} k_{\mu} p_{\nu} \right),
\]
comes back to haunt as IR singularity in the "nonplanar" diagram [4]. The effective
cutoff
\[
\Lambda_{\text{eff.}}^{-2} = \Lambda^{-2} + p \circ p, \quad \text{with} \quad p \circ p \equiv p_{\mu} \Theta^{\mu\rho} \Theta_{\rho\sigma} p_{\sigma},
\]
becomes finite as \(\Lambda \to \infty\), thus giving good UV behavior; while as \(p \to 0\), singularity
as IR reappears.

### 2.2 Anomalies in commutative gauge theory

Generally symmetries of an action, e.g.,
\[
I = \int \bar{\psi} \left( i\mathcal{D} \right) \psi,
\]
under
\[
\psi \to e^{i\alpha} \psi \quad \text{for } \alpha \text{ constant},
\]
or
\[
\psi \to e^{i\gamma\alpha} \psi \quad \text{for } \alpha \text{ constant},
\]
are violated upon quantization. In the path integral formulation
\[
Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-iI},
\]
anomalies in noncommutative gauge theories and $\mathcal{N} = 1$ superpotential

The anomaly is the consequence of noninvariance of the measure; while invariance of the action

$$\delta L = \alpha \partial_\mu j^\mu, \quad j^\mu = \bar{\psi} \gamma^\mu \psi, \quad \partial_\mu j^\mu = 0,$$

(same with $j_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$), leads to a conserved charge,

$$Q = \int j_0 d^3 x, \quad \dot{Q} = 0.$$  \hspace{1cm} (13)

To find the anomaly it is convenient to consider a modified derivation, where $\alpha$ is initially to depend on $x$,

$$\delta \psi = i\alpha(x)\psi(x),$$  \hspace{1cm} (14)

then

$$\delta I = \int \alpha(x) \partial_\mu j^\mu(x) = -\int \partial_\mu \alpha j^\mu = 0;$$

if $\alpha$ is to be a constant,

$$\partial_\mu j^\mu = 0.$$  \hspace{1cm} (15)

Under this change of variable, measure changes as

$$D\bar{\psi} D\psi \rightarrow D\bar{\psi} D\psi \exp\left(-2i \int \alpha(x) \sum_n \varphi_n^\dagger(x) \varphi_n(x)\right),$$  \hspace{1cm} (16)

where

$$\psi = \sum_n a_n \varphi_n, \quad i\slashed{D} \varphi_n = \lambda_n \varphi_n.$$  \hspace{1cm} (17)

Regularization of the sum,

$$\sum_n \varphi_n^\dagger \varphi_n \rightarrow \lim_{M \rightarrow \infty} \sum_n \varphi_n^\dagger \varphi_n \exp\left(-\frac{\lambda^2}{M^2}\right),$$  \hspace{1cm} (18)

replaces the exponent in (17) by a gauge invariant expression

$$\lim_{M \rightarrow \infty} \sum_n \varphi_n^\dagger \exp\left(\frac{(i\slashed{D})^2}{M^2}\right) \varphi_n,$$  \hspace{1cm} (19)

which gives no anomaly for the $U(1)$ symmetry

$$\partial_\mu j^\mu = 0,$$  \hspace{1cm} (20)

while giving an anomaly

$$\partial_\mu j_5^\mu = -\frac{g^2}{16\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu}, \quad \tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho},$$  \hspace{1cm} (21)

to the axial symmetry

$$\delta \psi = i\alpha \gamma_5 \psi \quad \text{axial (chiral).}$$  \hspace{1cm} (22)
2.3 Anomalies in noncommutative gauge theory

The same can now be repeated for the noncommutative gauge theory, where

\[ \mathcal{L} = \bar{\psi} \star (i \partial \gamma_5) \star \psi = \bar{\psi} \star (i \partial + ig A) \star \psi. \]  

(23)

The transformation

\[ \psi(x) \to e^{i \alpha_{\gamma_5}} \psi(x), \]  

(24)

leads to a classical symmetry,

\[ \Rightarrow \partial_\mu j_5^\mu = 0, \quad j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi, \]  

(25)

and

\[ Q_5 = \int j_5^5 \, d^3x, \quad \dot{Q}_5 = 0. \]  

(26)

To study its anomaly the modified procedure is used, which gives two distinct currents in contrast to ordinary theory,

a) the transformation

\[ \psi(x) \to e^{i \alpha_{\gamma_5}} \psi(x), \]  

(27)

leads to a covariantly conserved current \( J_5^\mu \):

\[ \Rightarrow J_5^\mu = \psi \star \bar{\psi} \left( \gamma^\mu \gamma_5 \right)_{\alpha\beta}, \quad D_\mu J_5^\mu = \partial_\mu J_5^\mu + ig [A_\mu, J_5^\mu] = 0. \]  

(28)

b) the transformation

\[ \psi(x) \to \psi \star e^{i \alpha_{\gamma_5}}, \]  

(29)

leads to an invariant conserved current \( j_5^5 \)

\[ \Rightarrow j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi, \quad \partial_\mu j_5^\mu = 0. \]  

(30)

Here \( \psi \) transforms in the fundamental representation \( \psi \to U \star \psi \) and \( \bar{\psi} \to \bar{\psi} \star U^\dagger \).

The measure of the path integral is again not invariant

\[ \mathcal{D} \psi \mathcal{D} \bar{\psi} \to \mathcal{D} \psi \mathcal{D} \bar{\psi} \ e^{-2i \int \alpha \ast A}, \]  

(31)

with \( A \) the corresponding anomaly.

Now the significant difference with the ordinary theory appears: The two choices for the change of variables lead to distinct anomalies for \( D_\mu J_5^\mu \) and \( \partial_\mu j_5^\mu \).

a) for \( J_5^5 \),

\[ \delta \psi = i \alpha_{\gamma_5} \psi(x), \]  

(32)

\[ A = \sum_n (\varphi_{\alpha})_{\beta} \star (\bar{\varphi}_{\alpha})_{\beta} \left( \gamma_5 \right)_{\alpha\beta}. \]  

(33)
Anomalies in noncommutative gauge theories and $\mathcal{N} = 1$ superpotential

After regularization,
\[
\mathcal{A} = \lim_{M \to \infty} \sum_n \left( e^{-i\mathcal{D}/M^2} \star \varphi_n \right)_\beta \left( \varphi^*_n \right)_\alpha (\gamma^5)_{\alpha\beta} \tag{34}
\]
and noting that $J_5^\mu$, $D_\mu J_5^\mu$, and $\mathcal{A}$ are covariant under gauge transformation,
\[
U : \mathcal{O} \to U \star \mathcal{O} \star U^{-1} \tag{35}
\]
a tedious calculation \cite{1, 2, 3} leads to
\[
D_\mu J_5^\mu = -\frac{g^2}{16\pi^2} F_{\mu\nu} \star \tilde{F}^{\mu\nu}; \tag{36}
\]
b) For $j_5^\mu$,
\[
\delta \psi = \psi(x) \star i\alpha(x)\gamma_5, \tag{37}
\]
\[
\mathcal{A} = \sum_n \varphi^*_n \star \gamma_5 \varphi_n, \tag{38}
\]
an invariant regularization is needed. Using Wilson line
\[
\mathcal{A} = \lim_{M \to \infty} \sum_n \varphi^*_n \star \gamma_5 \left[ e^{i\mathcal{D}/M^2} \right]_{\text{inv.}} \star \varphi_n(x), \tag{39}
\]
where
\[
\left[ \mathcal{O} \right]_{\text{inv.}} = \int d^k e^{ikx} \int dy \sum_n \frac{1}{n!} \prod_{i=1}^n \int d\sigma_i P_* \left[ W(y, \hat{k}) \mathcal{O}(y + \sigma_i \hat{k}) \right] \star e^{iky}, \tag{40}
\]
\[
W(y, \hat{k}) = \exp \left( i \int_0^1 d\sigma \hat{k} \cdot A(x + \sigma \hat{k}) \right), \tag{41}
\]
\[
\hat{k}_\mu = \Theta_\mu \nu k^\nu, \tag{42}
\]
\[
\mathcal{O} \to U \star \mathcal{O} \star U^{-1}, \quad \text{but} \tag{43}
\]
\[
\left[ \mathcal{O} \right] \to \left[ \mathcal{O} \right],
\]
on one gets for $\Theta p \to 0$
\[
\partial_\mu J_5^\mu = -\frac{g^2}{16\pi^2} F_{\mu\nu} \star \tilde{F}^{\mu\nu} \tag{44}
\]
with
\[
f(x) \star' g(x) \equiv f(x) \frac{\sin \left( \frac{1}{\pi} \frac{1}{\partial_\mu} \Theta^{\mu\nu} \frac{1}{\partial_\nu} \right)}{\frac{1}{\pi} \frac{1}{\partial_\mu} \Theta^{\mu\nu} \frac{1}{\partial_\nu}} g(x). \tag{45}
\]
While covariant $J_5^\mu$ anomaly involves the $\star$-product, the invariant $j_5^\mu$ anomaly involves the $\star'$-product \cite{2, 3}.
2.4 Konishi anomaly (noncommutative)

Similar considerations apply to the current of supersymmetric (SUSY) gauge theory. In $\mathcal{N} = 1$ SUSY gauge theory

$$ I = \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \Phi \star \Phi' + \int d^4x \, \cos \theta \, W_\alpha \star W^\alpha + \text{h.c.}, $$

with $\Phi(x,\theta)$ the hypermultiplet and $V(x,\theta,\bar{\theta})$ the gauge field,

$$ W_\alpha(x,\theta) = \bar{D}^2 \epsilon^\alpha \star D^\alpha \epsilon^V, $$

$$ D = \partial_\theta + i \left( \sigma^\mu \bar{\theta} \right) \partial_\mu; $$

the symmetry,

$$ \Phi \to e^{i\alpha(x,\theta)} \Phi, $$

$$ e^V \to e^{i\alpha} \star e^V \star e^{-i\alpha}, \quad \bar{D} \alpha = 0. $$

$$ J = \Phi \star \Phi' \star e^V, $$

$$ j = \tilde{\Phi} \star e^V \star \Phi $$

is anomalous. The anomaly is given by

$$ -\frac{1}{4} \bar{D}^2 J = -\frac{1}{32\pi^2} W_\alpha \star W^\alpha, $$

$$ -\frac{1}{4} \bar{D}^2 j = -\frac{1}{32\pi^2} W_\alpha \star W^\alpha + \cdots. $$

The calculation is the same as for nonSUSY gauge theory, except for the regulator $e^{L/M^2}$, where

$$ L = \frac{1}{16} \bar{D}^2 \epsilon^V \star D^2 \epsilon^V. $$

Lowest SUSY multiplet component of $L$ is $\mathcal{D}^2$, regulator brings SUSY and gauge invariance [3].

The regulator for $j$ is the same as for the nonSUSY case with $\mathcal{D}^2 \to L$ and a corresponding Wilson line.

### 3 Superpotential

It is known that effective superpotential of SUSY gauge theories is the sum of a perturbative and a nonperturbative part

$$ W_{\text{eff}}(S) = W_{\text{pert}} + W_{\text{non-pert}}, $$

$W_{\text{pert}}$ is calculated from Konishi anomaly, and $W_{\text{non-pert}}$ from the axial anomaly. Here

$$ S = W_\alpha \star W^\alpha $$

$$ S' = W_\alpha \star W^\alpha + \cdots. $$
The ellipses are the contribution of the Wilson line attachment.

To get $W_{\text{non-pert}}$, axial anomaly is used:

\begin{align*}
U_A(1) : \Phi(x, \theta) &\rightarrow e^{i\alpha} \Phi(x, \theta) \quad \text{and} \quad W_{\alpha}(x, \theta) \rightarrow W_{\alpha}(x, \theta), \\
U_R(1) : \Phi(x, \theta) &\rightarrow e^{i\alpha} \Phi(x, e^{-3i\alpha/2}) \quad \text{and} \quad W_{\alpha}(x, \theta) \rightarrow e^{3i\alpha/2} W_{\alpha}(x, e^{-3i\alpha/2})
\end{align*}

\( \Rightarrow \)

\[
\delta_A \mathcal{L} = \begin{cases} 
0, & \frac{p \circ p}{4} \gg \frac{1}{M^2}, \\
2N_f \alpha A', & \frac{p \circ p}{4} \ll \frac{1}{M^2}
\end{cases}
\]

(55)

and

\[
\delta_R \mathcal{L} = \begin{cases} 
2\alpha R(\lambda) A, & \frac{p \circ p}{4} \gg \frac{1}{M^2}, \\
2N_f \alpha R(\psi) A', & \frac{p \circ p}{4} \ll \frac{1}{M^2},
\end{cases}
\]

(56)

Here $A$ and $A'$ are the planar and nonplanar ABJ anomalies. They are defined by $A \equiv -\frac{1}{32\pi^2} F_{\mu\nu} \star \tilde{F}_{\mu\nu}$ and $A' \equiv -\frac{1}{32\pi^2} F_{\mu\nu} \star' \tilde{F}_{\mu\nu} + \cdots$ with the extra terms denoting the contribution of the open Wilson line.

In the "small" $\Theta p$ limit

\[
W_{\text{non-pert}}(T, S'; \Lambda N_f, \Lambda_\Theta) = S' \left( \log \left( \frac{S' N_f A_{\Theta}^{(N_f+6)}}{\Lambda_{N_f}^{2(3-N_f)}} \right) - N_f \right),
\]

(57)

with $\Lambda_\Theta = \Theta^{-1/2}$.

In the "large" $\Theta p$ limit

\[
W_{\text{dyn}}(S; \Lambda N_f, \Lambda_\Theta) = -S \left( \log \left( \frac{S A_{\Theta}^{3-2N_f}}{\Lambda_{N_f}^{2(3-N_f)}} \right) - 1 \right).
\]

(58)

Here $\Lambda_\Theta = \Theta^{-1/2}$, and $T_{ij} = \Phi_i \star \Phi_j$.

To get $W_{\text{pert}}$, choose

\[
W_{\text{tree}} = m T_{ii} + \lambda T_{ii}^2;
\]

(59)

and use Konishi anomaly to solve for $T$ in terms of $S'$ (small $\Theta p$ limit) and $S$ (large $\Theta p$ limit):
In the "small" $\Theta p$ limit

$$W_{\text{eff}} \left( S'; m, \lambda; \hat{\Lambda}_0, \hat{\Lambda}_\Theta \right) \bigg|_{|\sqrt{\Theta}p| \ll 1} =$$

$$= 6S' \log \frac{\hat{\Lambda}_\Theta}{\Lambda_0} - \frac{N_f}{2} S' - N_f \frac{m^2}{8\lambda} + (N_f^+ - N_f^-) \frac{m^2}{8\lambda} \sqrt{1 + \frac{8\lambda S'}{m^2}} +$$

$$+ S' \log \left[ \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^{N_f^+} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^{N_f^-} \right].$$

(60)

In the "large" $\Theta p$ limit

$$W_{\text{eff}} \left( \text{tr} T = \frac{-N_f m}{2\lambda}; m, \lambda; \hat{\Lambda}_0, \hat{\Lambda}_\Theta \right) \bigg|_{|\sqrt{\Theta}p| \gg 1} = -\frac{N_f m^2}{4\lambda} - S \left( \log \frac{S \hat{\Lambda}_\Theta^3}{\Lambda_0^6} - 1 \right).$$

(61)

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### References


