Non—generic symmetries and surface terms

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Integrable geometries were obtained by adding a total time derivative involving the components of the angular momentum to a given free Lagrangian. The motion on a sphere and its induced geometries are examined in details.

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1 Introduction

Killing–Yano tensors (KY) were introduced by Yano [1] from pure mathematical point of view [2] and the physical significance of these tensors was obtained by Gibbons and Holten [3]. A KY is an antisymmetric tensor define as

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0,$$

where $D_\lambda$ represents the covariant derivative. KY tensors of rank two are related to non-generic supersymmetries of the spinning particle model (see for more details Ref. [3]) and the geometrical duality depends on the existence of these tensors [4, 5]. Since KY tensors were introduced there were many attempts to applied them in various areas [6, 7, 8, 9, 10].

In this paper we made a link between the surface terms [11] and KY tensors and we review the results presented in [13].

The starting point is a given free Lagrangian $L(q^i, q^i)$ admitting a set of constants of motion denoted by $L_i$, $i = 1, \cdots, 3$. If we add the components of the angular momentum corresponding to $L_i$, the extended Lagrangian [12]

$$L' = L + \lambda^i L_i, \quad i = 1, \cdots, 3$$

becomes $L' = \frac{1}{2} a_{ij} q^i q^j$. In this context the second term in (2) is a total time derivative and the Lagrangians $L$ and $L'$ are equivalent. We mention that the matrix $a_{ij}$ is symmetric by construction. The next step is to find whether $a_{ij}$ is singular or not. Assuming that $a_{ij}$ is a singular $n \times n$ matrix of rank $n - 1$ we obtain non-singular symmetric matrices of order $(n - 1) \times (n - 1)$, where $n$ will be 3, 5 and 6. Finally we consider the obtained matrices as metrics on the extended space and we investigate their Killing vectors and KY tensors.
2 Angular momentum and Killing–Yano tensors

The Lagrangian to start with is
\[ L_0 = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + \dot{\lambda}_3 (x \dot{y} - y \dot{x}), \]
which in the compact notation becomes \( L' = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j \). Here \( a_{ij} \) is given by
\[ a_{ij} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix}. \] (4)
The metric (4) admits the Killing vector \( V = (y, -x, 0) \).

Solving (1) for (4) we obtained the following KY tensor
\[ f_{12} = 0, \quad f_{23} = -Cx \sqrt{x^2 + y^2}, \quad f_{13} = Cy \sqrt{x^2 + y^2}, \] (5)
where \( C \) represents a constant [13].

As it known a KY tensor of rank two generates a Killing tensor as
\[ K_{ij} = f_{\mu \lambda} f^{\lambda}_{ij}. \] (6)
In our case, using (5) and (6) a Killing tensor is constructed as
\[ K_{ij} = \begin{pmatrix} y^2 & -xy & -y(y^2 + x^2) \\ -xy & x^2 & x(x^2 + y^2) \\ -y(y^2 + x^2) & x(x^2 + y^2) & 0 \end{pmatrix}. \] (7)
The second step is to add two components of the angular momentum to a free, three-dimensional Lagrangian. The corresponding extended Lagrangian becomes
\[ L' = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \dot{\lambda}_1 (y \dot{z} - z \dot{y}) + \dot{\lambda}_2 (z \dot{x} - x \dot{z}) \] (8)
and from (8) we obtain \( a_{ij} \) as the following non-singular matrix
\[ a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & y & -x \\ 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \end{pmatrix}. \] (9)
The metric (9) admits three Killing vectors as
\[ V_1 = (y, -x, 0, 0, 0), \quad V_2 = (0, -z, y, 0, 0), \quad V_3 = (z, 0, -x, 0, 0). \] (10)
For metric (10) KY tensors components are as follows
\[ f_{15} = -Gxy, \quad f_{14} = G(z^2 + y^2), \]
\[ f_{24} = -Gxy, \quad f_{34} = -Gxz, \]
\[ f_{25} = G(x^2 + z^2), \quad f_{35} = \frac{-Gzxy}{x}, \]
\[ f_{12} = Cz, \quad f_{13} = -Cy, \] (11)
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others zero. Here $C$ and $G$ are constants. The corresponding Killing tensor has the following form

$$K = \begin{pmatrix} G(-2C+G)(z^2+y^2) & GD_{xy} & GD_{zx} & 0 & G^2r^2z \\ GD_{xy} & -GD(x^2+z^2) & GD_{zy} & -r^2zG^2 & 0 \\ GD_{zx} & GD_{zy} & -GD(y^2+x^2) & G^2r^2y & -G^2r^2x \\ 0 & -G^2zr^2 & G^2yr^2 & 0 & 0 \\ G^2zr^2 & 0 & -G^2xr^2 & 0 & 0 \end{pmatrix}.$$  

(12)

where $D = 2C + G$ and $r^2 = x^2 + y^2 + z^2$.

If we add all angular momentum components to the Lagrangian of the free particle in three-dimensions, the extended Lagrangians $L'$ is given by

$$L' = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \lambda_1(y\dot{z} - z\dot{y}) + \lambda_2(z\dot{x} - x\dot{z}) + \lambda_3(x\dot{y} - y\dot{x}).$$  

(13)

In compact form (13) has the form $L' = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$. Here $a_{ij}$ is singular matrix given by

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \\ 0 & -z & y & 0 & 0 & 0 \\ z & 0 & -x & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

Using the fact that the rank of (14) is 5 we obtained three non-singular symmetric matrices corresponding to three non-zero minors. The first one is given by (9) and the other two are as

$$b^{(2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & -y \\ 0 & 1 & 0 & -z & x \\ 0 & 0 & 1 & y & 0 \\ 0 & -z & y & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}.$$  

(15)

and

$$b^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & z & -y \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & -x & 0 \\ z & 0 & -x & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \quad (16)$$

By direct calculations [13] we obtain that (15) and (16) admit three Killing vectors given by (10) and a KY tensor possessing the following non-zero components

$$f_{12} = z, \quad f_{13} = -y, \quad f_{23} = x. \quad (17)$$
3 Induced geometries on a sphere

The motion on a sphere admits four constants of motion, the Hamiltonian and three components of the angular momentum [14]. The aim of this section is to use the surface terms and to generate four-dimensional manifolds. The Lagrangian to start with is given by

\[
L' = \frac{1}{2} \left( 1 + \frac{x^2}{u} \right) \dot{x}^2 + \frac{1}{2} \left( 1 + \frac{y^2}{u} \right) \dot{y}^2 + \frac{xy}{u} \dot{x} \dot{y} - \frac{xy}{\sqrt{u}} \dot{\lambda}_1 \dot{x} + \left( \frac{x^2}{\sqrt{u}} + \sqrt{u} \right) \dot{\lambda}_2 \dot{x} - \left( \frac{y^2}{\sqrt{u}} + \sqrt{u} \right) \dot{\lambda}_1 \dot{y} + \frac{xy}{\sqrt{u}} \dot{\lambda}_2 \dot{y} + x \dot{\lambda}_3 \dot{y} - y \dot{\lambda}_3 \dot{x},
\]

where \( u = 1 - x^2 - y^2 \). Using (18) we identify the singular matrix \( a_{ij} \) as

\[
a_{ij} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & \frac{xy}{u} & \frac{x^2}{\sqrt{u}} + \sqrt{u} & -y \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & \frac{xy}{\sqrt{u}} & x \\
-\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & 0 & 0 & 0 \\
\frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 & 0 \\
-\frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{u} & x & 0 & 0 
\end{pmatrix}.
\]

Using the fact that (19) is a singular matrix of rank 4 we identify three symmetric minors of order four. If we consider these minors as a metric we observed that they are not conformally flat but their scalar curvatures are zero.

The first metric is given by

\[
y^{(1)}_{\mu\nu} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & \sqrt{u} + \frac{x^2}{\sqrt{u}} & -y \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\
\sqrt{u} + \frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\
-\frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{u} & x & 0 & 0 
\end{pmatrix}.
\]

The Killing vectors of (20) are given by [13]

\[
V_1 = (y, -x, 0, 0), \\
V_2 = \left( \sqrt{1 - x^2 - y^2} + \frac{x^2}{1 - x^2 - y^2}, \frac{xy}{1 - x^2 - y^2}, 0, 0 \right), \\
V_3 = \left( -\frac{xy}{1 - x^2 - y^2}, -\sqrt{1 - x^2 - y^2} - \frac{y^2}{1 - x^2 - y^2}, 0, 0 \right).
\]
The next step is to investigate its KY tensors. Solving (1) we obtain the following set of solutions:

a. One solution is \( f_{21} = \frac{C_1}{\sqrt{1 - x^2 - y^2}} \), others zero.

b. Two-by-two solution has the form: \( f_{31} = f_{42} = C \).

c. Three by three solution is \( f_{21} = \frac{C_1}{\sqrt{-1 + x^2 + y^2}} \) and \( f_{31} = f_{42} = C \), where \( C \) and \( C_1 \) are constants.

From (18) another two metrics can be identified as

\[
g^{(2)}_{\mu\nu} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & - \frac{xy}{\sqrt{u}} & - y \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & - \sqrt{u} - \frac{y^2}{\sqrt{u}} & x \\
- \frac{xy}{\sqrt{u}} & - \frac{y^2}{\sqrt{u}} & 0 & 0 \\
- y & x & 0 & 0
\end{pmatrix}
\] (22)

and

\[
g^{(3)}_{\mu\rho} = \begin{pmatrix}
1 + \frac{x^2}{u} & \frac{xy}{u} & - \frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}} + \sqrt{u} \\
\frac{xy}{u} & 1 + \frac{y^2}{u} & - \frac{y^2}{\sqrt{u}} - \sqrt{u} & \frac{xy}{\sqrt{u}} \\
- \frac{xy}{\sqrt{u}} & - \frac{y^2}{\sqrt{u}} & 0 & 0 \\
\frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0
\end{pmatrix}
\] (23)

By direct calculations we obtained that (22) and (23) have the same Killing vector as in (21). Solving (1) for (22) and (23) we find one non-zero component of KY tensor as follows

\[ f_{21} = \frac{C_1}{\sqrt{1 - x^2 - y^2}}. \] (24)

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References