

# The representation theory of the Heisenberg group and beyond

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In this paper we present some recent and new developments in the theory of  $p$ -mechanics.  $p$ -Mechanics is a consistent physical theory which contains both classical and quantum mechanics. The Heisenberg group and its representation theory is the basis of  $p$ -mechanics. We give a summary of recent results on  $p$ -mechanical observables, states and canonical transformations. In doing so we exhibit relations between the quantum and classical image of these objects. We also present some new work on the Kepler/Coulomb problem. This involves constructing a new Hilbert space which represents the dynamics of the Kepler/Coulomb problem in a simple form.

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## 1 Introduction

Since the time of von Neumann the infinite dimensional Schrödinger representation of the Heisenberg group has been used in the mathematical formulation of quantum mechanics. By the Stone–von Neumann theorem all unitary irreducible infinite dimensional representations of the Heisenberg group are unitarily equivalent to the Schrödinger representation. There is also a system of one dimensional representations of the Heisenberg group — these are often ignored and just included in the Stone–von Neumann theorem for mathematical completeness. In [3, 4] it is shown that these one dimensional representations play the same role in classical mechanics which the infinite dimensional representations play in quantum mechanics.  $p$ -Mechanics uses both the one and infinite dimensional representations to obtain relations between classical and quantum mechanics. It shows that both classical and quantum mechanics are derived from the same source separated by the one and infinite dimensional representations respectively.

In Section 2 we present some background material on the representation theory of the Heisenberg group.  $p$ -Mechanical observables and their time evolution is the subject of Section 3. Section 4 contains a description of the role of states in  $p$ -mechanics. In Section 5 we describe the representation of both linear and non-linear canonical transformations in  $p$ -mechanics. In Section 6 we present some new results on the  $p$ -mechanical Kepler/Coulomb problem. It is shown that the current form of  $p$ -mechanics is insufficient for analysing the Kepler/Coulomb problem. We use Klauder’s coherent states for the Kepler/Coulomb problem to derive a new Hilbert space which is particularly useful in studying the Kepler/Coulomb problem. The dynamics for the Kepler/Coulomb problem take a particularly nice form in this

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Hilbert space. This Hilbert space is limited as it can only deal with a subset of observables and states.

## 2 The Heisenberg group and its representations

In this section we present some background material on the Heisenberg group and its representations.

**Definition 2.1.** *The Heisenberg group [1, 2] (denoted  $\mathbb{H}^n$ ) is the set of all triples in  $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$  under the law of multiplication*

$$(s, x, y) \cdot (s', x', y') = \left( s + s' + \frac{1}{2}(x \cdot y' - x' \cdot y), x + x', y + y' \right). \quad (1)$$

The non-commutative convolution of two functions  $B_1, B_2 \in L^1(\mathbb{H}^n)$  is defined as

$$(B_1 * B_2)(g) = \int_{\mathbb{H}^n} B_1(h)B_2(h^{-1}g)dh = \int_{\mathbb{H}^n} B_1(gh^{-1})B_2(h)dh,$$

where  $dh$  is Harr Measure on  $\mathbb{H}^n$ , which is just Lebesgue measure on  $\mathbb{R}^{2n+1}$ ,  $ds dx dy$ . This operation can be extended to the convolution of two distributions in the natural way — see [2, Chap. 0]. The most common representation of the Heisenberg group is the Schrödinger representation. The Schrödinger representation [1, Sect. 1.3] for  $h > 0$  is defined on  $L^2(\mathbb{R}^n)$  as

$$(\rho_h^S(s, x, y)\psi)(\xi) = e^{-2\pi ihs + 2\pi i x \xi + \pi i h x y} \psi(\xi + hy). \quad (2)$$

It has been shown that this representation is unitary [1, Sect. 1.3], and irreducible [1, Prop. 1.43]. In this paper we only briefly look at this infinite dimensional representation; instead we concentrate on other forms of the infinite dimensional representation and also the often neglected family of one dimensional representations.

**Definition 2.2.** *We define the space  $F^2(\mathcal{O}_h)$  [3, 4] as*

$$F^2(\mathcal{O}_h) = \{f_h(q, p) \in L^2(\mathbb{R}^{2n}) : D_h^j f_h = 0, \text{ for } 1 \leq j \leq n\}, \quad (3)$$

where the operator  $D_h^j$  on  $L^2(\mathbb{R}^{2n})$  is defined as  $\frac{h}{2} \left( \frac{\partial}{\partial p_j} + i \frac{\partial}{\partial q_j} \right) + 2\pi(p_j + iq_j)$ .

An inner product on  $F^2(\mathcal{O}_h)$  is given by

$$\langle v_1, v_2 \rangle_{F^2(\mathcal{O}_h)} = \left( \frac{4}{h} \right)^n \int_{\mathbb{R}^{2n}} v_1(q, p) \overline{v_2(q, p)} dq dp. \quad (4)$$

$F^2(\mathcal{O}_h)$  is a Hilbert space with respect to this inner product [5, Sect. 4.1]. The representation  $\rho_h$  [3, 4] of  $\mathbb{H}^n$  on  $F^2(\mathcal{O}_h)$  is defined by

$$\rho_h(s, x, y) : f_h(q, p) \mapsto e^{-2\pi i(hs + qx + py)} f_h \left( q - \frac{h}{2}y, p + \frac{h}{2}x \right), \quad (5)$$

which is unitary with respect to the inner product defined in (4). The crucial theorem which motivates the whole of  $p$ -mechanics is

**Theorem 2.3.** (The Stone–von Neumann Theorem) All unitary irreducible representations of the Heisenberg group,  $\mathbb{H}^n$ , up to unitary equivalence, are either:

- (i) of the form  $\rho_h$  on  $F^2(\mathcal{O}_h)$  from equation (5), or
- (ii) for  $(q, p) \in \mathbb{R}^{2n}$  the commutative one–dimensional representations on  $\mathbb{C} = L^2(\mathcal{O}_{(q,p)})$

$$\rho_{(q,p)}(s, x, y)u = e^{-2\pi i(q \cdot x + p \cdot y)}u. \quad (5)$$

For a proof of this theorem see [6].

### 3 Observables and their dynamics in $p$ –mechanics

The basic idea of  $p$ –mechanics is to choose particular functions or distributions on  $\mathbb{H}^n$  which under the infinite dimensional representation will give quantum mechanical observables, while under the one dimensional representation will give classical mechanical observables. In doing this it is shown that both mechanics are derived from the same source.  $p$ –Mechanical observables can be realized as operators (some of which are unbounded) on a subset of  $L^2(\mathbb{H}^n)$  generated by convolutions of the chosen functions or distributions. To define  $p$ –mechanical observables properly we need to introduce a map from the set of classical observables to the set of  $p$ –mechanical observables. In [3,4] a map of  $p$ –mechanization,  $\mathcal{P}$ , from the set of classical observables to the set of  $p$ –mechanical observables is defined as

$$(\mathcal{P}f)(s, x, y) = \delta(s)\check{f}(x, y), \quad (6)$$

where  $f$  is any classical observable and  $\check{f}$  is the inverse Fourier transform of  $f$  (that is,  $\check{f}(x, y) = \int_{\mathbb{R}^{2n}} f(q, p)e^{2\pi i(qx + py)} dq dp$ ).

**Definition 3.1 ( $p$ –Mechanical Observables).** The set of  $p$ –mechanical observables is the image of the set of classical observables under the map  $\mathcal{P}$  from equation (6).

Clearly this definition depends on how the set of classical observables is defined. Any physically reasonable classical mechanical observable can be realized as an element of  $\mathcal{S}'(\mathbb{R}^{2n})$  (see [10] for a definition of  $\mathcal{S}'$ ). Since the Fourier transform maps  $\mathcal{S}'(\mathbb{R}^{2n})$  into itself,  $\mathcal{S}'(\mathbb{H}^n)$  is a natural choice for the set of  $p$ –mechanical observables. It includes the image of all classical observables which are polynomials or exponentials of the variables  $q$  and  $p$ .

If we take the  $\rho_h$  representation (5) of many of the distributions described above we would get unbounded operators. For example the  $p$ –mechanisation of the classical position coordinate  $q$  is the distribution  $\frac{\partial}{\partial x}\delta(s)\delta(x)\delta(y)$ ; under the  $\rho_h$  representation this will become the unbounded operator  $\frac{\hbar}{2}\frac{\partial}{\partial p} - 2\pi iqI$ . This operator is clearly not defined on the whole of  $F^2(\mathcal{O}_h)$ . This technical problem can be solved by the usual method of rigged Hilbert spaces (also known as Gelfand triples) [9] which uses the theory of distributions. Another approach to dealing

with unbounded operators is given by using the Gårding space as explained in [1, Chap. 0].

The dynamics of a  $p$ -mechanical system are described in [3, 4, 7] using the universal brackets (also known as the  $p$ -mechanical brackets). Before we can define the universal brackets we need to define the operator  $\mathcal{A}$ .

**Definition 3.2.** *Let  $S$  be the operator  $\frac{\partial}{\partial s}$  on  $C^\infty(\mathbb{H}^n)$ . The operator  $\mathcal{A}$  is defined on exponents by*

$$S\mathcal{A} = 4\pi^2 I, \quad \text{where} \quad \mathcal{A}e^{2\pi ihs} = \begin{cases} \frac{2\pi}{ih} e^{2\pi ihs}, & \text{if } h \neq 0, \\ 4\pi^2 s, & \text{if } h = 0, \end{cases} \quad (7)$$

and can be extended by linearity to the whole of  $L^1(\mathbb{H}^n)$ . It can also be extended by linearity to the whole of  $\mathcal{S}(\mathbb{H}^n)$  and then the adjoint is an operator on  $\mathcal{S}'(\mathbb{H}^n)$ .  $\mathcal{A}$  is called the anti-derivative operator since it is a right inverse to  $\frac{\partial}{\partial s}$ .

**Definition 3.3.** *The universal brackets (also known as  $p$ -mechanical brackets) are*

$$\{\{B_1, B_2\}\} = (B_1 * B_2 - B_2 * B_1)\mathcal{A}. \quad (8)$$

It is shown in [7, Prop. 3.5] that under the one and infinite dimensional representations the universal brackets become the Poisson brackets and the quantum commutator respectively. Hence for a system with Hamiltonian  $B_H$  (the  $p$ -mechanisation of the classical Hamiltonian  $H$ ) solving the  $p$ -dynamic equation

$$\frac{dB}{dt} = \{\{B, B_H\}\} \quad (9)$$

will give the quantum and classical dynamics under the infinite and one dimensional representations respectively.

**Example 3.4 (The Harmonic Oscillator).** The classical Hamiltonian of the harmonic oscillator with frequency  $\omega$  and mass  $m$  is

$$H(q, p) = \frac{1}{2} \left( m\omega^2 q^2 + \frac{1}{m} p^2 \right). \quad (10)$$

This is a  $C^\infty$  function which can be realized as an element of  $\mathcal{S}'(\mathbb{R}^{2n})$ . The  $p$ -mechanisation (see equation (6)) of this is the  $p$ -mechanical harmonic oscillator Hamiltonian<sup>1)</sup>

$$B_H(s, x, y) = -\frac{1}{8\pi^2} \left( m\omega^2 \delta(s) \delta^{(2)}(x) \delta(y) + \frac{1}{m} \delta(s) \delta(x) \delta^{(2)}(y) \right). \quad (11)$$

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<sup>1)</sup>  $\delta(s) \delta^{(2)}(x) \delta(y)$  is used to denote the distribution  $\frac{\partial^2}{\partial x^2} \delta(s) \delta(x) \delta(y)$ .

In [3, Eq. 4.14] it is shown that the  $p$ -dynamic equation (9) for an arbitrary  $p$ -mechanical observable in this system is

$$\frac{dB}{dt} = \{B, B_H\} = \omega^2 m y \frac{\partial B}{\partial x} - \frac{x}{m} \frac{\partial B}{\partial y},$$

which has solution

$$B(t; s, x, y) = B_0 \left( s, x \cos(\omega t) + m\omega y \sin(\omega t), -\frac{1}{m\omega} x \sin(\omega t) + y \cos(\omega t) \right). \quad (12)$$

**Example 3.5 (The Forced Oscillator).** The classical Hamiltonian for an oscillator of frequency  $\omega$  and mass  $m$  being forced by a real function of a real variable  $z(t)$  is

$$H(t, q, p) = \frac{1}{2} \left( m\omega^2 q^2 + \frac{1}{m} p^2 \right) - z(t)q.$$

Through the procedure of  $p$ -mechanisation (see equation (6)) we get the  $p$ -mechanical forced oscillator Hamiltonian to be

$$B_H(t, s, x, y) = -\frac{1}{8\pi^2} \left( m\omega^2 \delta(s) \delta^{(2)}(x) \delta(y) + \frac{1}{m} \delta(s) \delta(x) \delta^{(2)}(y) \right) - \frac{z(t)}{2\pi i} \delta(s) \delta^{(1)}(x) \delta(y). \quad (13)$$

In [8] it is shown that the  $p$ -dynamic equation for the forced oscillator is

$$\frac{dB}{dt} = \omega^2 m y \frac{\partial B}{\partial x} - \frac{x}{m} \frac{\partial B}{\partial y} - 2\pi i y z(t) B, \quad (14)$$

which has solution

$$B(t; s, x, y) = \exp \left[ -2\pi i \left( \frac{1}{m\omega} \int_0^t z(\tau) \sin(\omega\tau) d\tau X(t) + \int_0^t z(\tau) \cos(\omega\tau) d\tau Y(t) \right) \right] B(0; s, X(t), Y(t)), \quad (15)$$

where

$$X(t) = x \cos(\omega t) + m\omega y \sin(\omega t),$$

$$Y(t) = -\frac{x}{m\omega} \sin(\omega t) + y \cos(\omega t).$$

#### 4 States in $p$ -mechanics

We now describe the role states play in  $p$ -mechanics. States in  $p$ -mechanics are functionals on the set of  $p$ -mechanical observables and come in two forms — elements of a Hilbert space and integration kernels.

**Definition 4.1.** The Hilbert space  $\mathcal{H}_h^2$ ,  $h \in \mathbb{R} \setminus \{0\}$ , is defined as the set of functions on  $\mathbb{H}^n$

$$\mathcal{H}_h^2 = \left\{ e^{2\pi i h s} f(x, y) : E_h^j f = 0 \quad 1 \leq j \leq n \quad \text{and} \quad f \in L^2(\mathbb{R}^{2n}) \right\}, \quad (16)$$

where the operator  $E_h^j = \pi h(y - ix) + i \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$  (this is the Fourier transform of  $D_h^j$  from (3)).

The inner product on  $\mathcal{H}_h^2$  is defined as

$$\langle v_1, v_2 \rangle_{\mathcal{H}_h^2} = \left( \frac{4}{h} \right)^n \int_{\mathbb{R}^{2n}} v_1(s, x, y) \overline{v_2(s, x, y)} \, dx \, dy. \quad (17)$$

The set of  $p$ -mechanical observables acts on  $\mathcal{H}_h^2$  by convolution. For many observables this will give rise to unbounded operators which are not defined on the whole of  $\mathcal{H}_h^2$ . This problem is solved as before by the use of rigged Hilbert spaces. It is shown in [8, Eq. 3.4] that any element  $v \in \mathcal{H}_h^2$  is of the form  $v(s, x, y) = e^{2\pi i h s} \hat{f}(x, y)$  for some  $f \in F^2(\mathcal{O}_h)$  ( $\hat{f}$  denotes the Fourier transform of  $f$ ). The following result from [8] shows that states in  $\mathcal{H}_h^2$  and  $F^2(\mathcal{O}_h)$  will give the same expectation values.

**Theorem 4.2.** If  $B$  is a  $p$ -mechanical observable and  $v_1, v_2 \in \mathcal{H}_h^2$  are of the form

$$\begin{aligned} v_1(s, x, y) &= e^{2\pi i h s} \hat{f}_1(x, y), \\ v_2(s, x, y) &= e^{2\pi i h s} \hat{f}_2(x, y), \end{aligned} \quad (18)$$

where  $f_1$  and  $f_2$  are in  $F^2(\mathcal{O}_h)$ , then we have the relationship

$$\langle B * v_1, v_2 \rangle_{\mathcal{H}_h^2} = \langle \rho_h(B) f_1, f_2 \rangle_{F^2(\mathcal{O}_h)}. \quad (19)$$

Each of these states can also be realized by an appropriate integration kernel.

**Theorem 4.3.** If  $l(s, x, y)$  is defined to be the kernel

$$l(s, x, y) = \left( \frac{4}{h} \right)^n \int_{\mathbb{R}^{2n}} \overline{v((s, x, y)^{-1}(s', x', y'))} v((s', x', y')) \, dx' \, dy', \quad (20)$$

then

$$\langle B * v, v \rangle_{\mathcal{H}_h^2} = \int_{\mathbb{H}^n} B(s, x, y) \overline{l(s, x, y)} \, ds \, dx \, dy.$$

Theorem 4.3 is proved in [8]. Next we define the time evolution of the  $\mathcal{H}_h^2$  states. The operator  $\mathcal{A}$  on  $v \in \mathcal{H}_h^2$  is just  $\mathcal{A}v = \frac{2\pi}{i\hbar} v$  which is a left and right inverse of  $\frac{1}{4\pi^2} \frac{\partial}{\partial s}$ .

**Definition 4.4.** *If we have a system with energy  $B_H$  then an arbitrary vector  $v \in \mathcal{H}_h^2$  evolves under the equation*

$$\frac{dv}{dt} = B_H * \mathcal{A}v = \mathcal{A}B_H * v. \quad (21)$$

The operation of left convolution preserves each  $\mathcal{H}_h^2$  so this time evolution is well defined. Equation (21) implies that if we have  $B_H$  time-independent then for any  $v \in \mathcal{H}_h^2$

$$v(t; s, x, y) = e^{tB_H \mathcal{A}} v(0; s, x, y),$$

where  $e^{B_H \mathcal{A}}$  is the exponential of the operator of applying  $\mathcal{A}$  and then applying the left convolution of  $B_H$  — this operator is defined using Stone's theorem [11, Sect. 8.4].

**Theorem 4.5.** *If we have a system with energy  $B_H$  (assumed to be Hermitian) then for any state  $v \in \mathcal{H}_h^2$  and any observable  $B$*

$$\frac{d}{dt} \langle B * v, v \rangle = \langle \{[B, B_H] * v, v \rangle.$$

We now define the time evolution of kernel states. Before we can do this we need to define the concept of kernel self-adjointness.

**Definition 4.6.** *A  $p$ -mechanical observable,  $B$ , is said to be kernel self-adjoint if the adjoint of the operator  $\{[\cdot, B]\}$  on the set of  $p$ -mechanical observables is the operator  $\{[B, \cdot]\}$  on the set of kernels (which are functionals on the set of  $p$ -mechanical observables). This is equivalent to the following equation holding*

$$\langle \{[C, B]\}, l \rangle = \langle C, \{[B, l]\} \rangle$$

for any  $p$ -mechanical observable  $C$ , where the brackets  $\langle \cdot, \cdot \rangle$  represent

$$\langle B, l \rangle = \int B(g) \overline{l(g)} dg.$$

The  $p$ -mechanical position and momentum observables are both kernel self adjoint and so are the  $p$ -mechanical Hamiltonians for the forced and harmonic oscillators. Hence all the Hamiltonians considered in this paper are kernel self adjoint.

**Definition 4.7.** *If we have a system with a kernel self-adjoint  $p$ -mechanical Hamiltonian,  $B_H$ , then an arbitrary kernel  $l \in \mathcal{L}_h$ ,  $h \in \mathbb{R}$ , evolves under the equation*

$$\frac{dl}{dt} = \{[B_H, l]\}. \quad (22)$$

The next theorem from [8] shows that the time evolution of these kernels coincides with the time evolution of  $p$ -mechanical observables.

**Theorem 4.8.** *If  $l$  is a kernel evolving under equation (22) then any observable  $B$  will satisfy*

$$\frac{d}{dt} \int_{\mathbb{H}^n} B \bar{l} dg = \int_{\mathbb{H}^n} \{[B, B_H]\} \bar{l} dg.$$

In [8] an overcomplete system of coherent states in  $\mathcal{H}_h^2$  was derived using representations of the Heisenberg group

$$\begin{aligned} v_{(h,a,b)}(s, x, y) &= \\ &= v_{(h,0,0)}\left(s + \frac{1}{2h}(by + ax), x + \frac{b}{h}, y - \frac{a}{h}\right) = \\ &= \exp\left(2\pi ihs + \pi i(by + ax) - \frac{\pi h}{2}\left(x + \frac{b}{h}\right)^2 - \frac{\pi h}{2}\left(y - \frac{a}{h}\right)^2\right). \end{aligned} \quad (23)$$

The corresponding kernel coherent states are

$$l_{(h,a,b)} = \exp\left(2\pi ihs + 2\pi i(ax + by) - \frac{\pi h}{2}(x^2 + y^2)\right). \quad (24)$$

It is shown in [8] that if we choose  $B = \mathcal{P}(f)$  then

$$\lim_{h \rightarrow 0} \langle B * v_{(h,q,p)}, v_{(h,q,p)} \rangle = \int_{\mathbb{H}^n} B \overline{l_{(h,q,p)}} dg = f(q, p).$$

By the usual theory of coherent states in a Hilbert space any element  $v \in \mathcal{H}_h^2$  can be written as

$$v = \int_{\mathbb{R}^{2n}} \langle v, v_{(h,q,p)} \rangle v_{(h,q,p)} dq dp. \quad (25)$$

## 5 Canonical transformations in $p$ -mechanics

In this chapter we discuss the representation of canonical transformations in  $p$ -mechanics. A canonical transformation in classical mechanics is a map  $A$  defined on phase space which preserves the symplectic form<sup>2)</sup>, that is

$$\omega(A(q, p), A(q', p')) = \omega((q, p), (q', p')). \quad (26)$$

If  $A$  is a linear canonical transformation then its effect on a  $p$ -mechanical observable  $B$  is just [3]

$$B(s, x, y) \mapsto B(s, (A^{-1})^*(x, y)).$$

The image of non-linear canonical transformations in  $p$ -mechanics is a much more delicate subject — this is the main focus of [6]. The approach we take continues the work of Moshinsky and his collaborators. Our aim is to understand the operator  $U$  on  $\mathcal{H}_h^2$  which corresponds to the classical canonical transformation.

<sup>2)</sup>  $\omega$  is defined as  $\omega((q, p), (q', p')) = qp' - q'p$ .



Our method starts with the observation that a canonical transformation in classical mechanics described by  $2n$  independent relations

$$q_i \rightarrow Q_i(q, p), \quad p_i \rightarrow P_i(q, p), \quad (27)$$

$i = 1, \dots, n$  where  $\{Q_i, P_j\}_{q,p} = \delta_{ij}$  can be realized implicitly by  $2n$  functional relations

$$f_i(q, p) = F_i(Q, P), \quad g_i(q, p) = G_i(Q, P), \quad (28)$$

$i = 1, \dots, n$  where  $\{f_i, g_j\}_{q,p} = \{F_i, G_j\}_{Q,P}$ . The advantage of describing the canonical transformation implicitly is that the  $p$ -mechanisation (see equation (6)) of the functions in (28) may be easier to define than the functions in equations (27).

In Dirac's original treatment of quantum canonical transformations [18] he proposed that the canonical transformation from equations (27) should be represented in quantum mechanics by an unitary operator  $U$  on a Hilbert space such that

$$\tilde{Q}_i = U \tilde{q}_i U^{-1} \quad \text{and} \quad \tilde{P}_i = U \tilde{p}_i U^{-1},$$

$i = 1, \dots, n$ . Here  $\tilde{Q}_i, \tilde{P}_i, \tilde{q}_i, \tilde{p}_i$  are the quantum mechanical observables corresponding to the classical mechanical observables  $Q_i, P_i, q_i, p_i$  respectively.

In [15] Mello and Moshinsky suggested that in some circumstances it is easier to define the operator  $U$  by the equations

$$\tilde{F}U = U\tilde{f} \quad \text{and} \quad \tilde{G}U = U\tilde{g},$$

where  $\tilde{F}, \tilde{G}, \tilde{f}, \tilde{g}$  are the operators corresponding to the classical observables  $F, G, f, g$  from equations (28).

We proceed to transfer this approach into  $p$ -mechanics. We want to understand the operator  $U$  which is defined by the equations

$$U\mathcal{P}(f_i(q, p)) * v = \mathcal{P}(F_i(Q, P)) * Uv, \quad (29)$$

$$U\mathcal{P}(g_i(q, p)) * v = \mathcal{P}(G_i(Q, P)) * Uv, \quad (30)$$

where  $\mathcal{P}$  is the map of  $p$ -mechanisation (see equation (6)) and  $v$  is any element of  $\mathcal{H}_h^2$ . We wish to find out the matrix elements of the operator  $U$  with respect to the  $\mathcal{H}_h^2$  coherent states from equation (23), that is  $\langle Uv_{(h,a,b)}, v_{(h,a',b')} \rangle$ . In [6] it is shown that  $m(a, b, a', b') = \langle Uv_{(h,a,b)}, v_{(h,a',b')} \rangle$  satisfies the following integral equation

$$\int_{\mathbb{R}^{2n}} m(a'', b'', a', b') \langle \mathcal{P}(f_i) * v_{(h,a,b)}, v_{(h,a'',b'')} \rangle da'' db'' = \quad (31)$$

$$= \int_{\mathbb{R}^{2n}} m(a, b, a'', b'') \langle \mathcal{P}(F_i) * v_{(h,a'',b'')}, v_{(h,a',b')} \rangle da'' db'',$$

$$\int_{\mathbb{R}^{2n}} m(a'', b'', a', b') \langle \mathcal{P}(g_i) * v_{(h,a,b)}, v_{(h,a'',b'')} \rangle da'' db'' = \quad (32)$$

$$= \int_{\mathbb{R}^{2n}} m(a, b, a'', b'') \langle \mathcal{P}(G_i) * v_{(h,a'',b'')}, v_{(h,a',b')} \rangle da'' db''.$$

By equation (25) if we know  $m(a, b, c, d)$  then we know the effect of  $U$  on any state in  $\mathcal{H}_h^2$ .

## 6 The Kepler/Coulomb problem

### 6.1 $p$ -mechanisation of the Kepler/Coulomb problem

In this section we look at the Kepler/Coulomb problem in detail. The Kepler/Coulomb Hamiltonian in three dimensional classical mechanics is<sup>3)</sup>

$$H(q, p) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|}. \quad (33)$$

All the norms in the above equation are the 2-norm on  $\mathbb{R}^3$  (that is  $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ ). Using the fact that the inverse Fourier transform of  $\|q\|^{-1}$  is the element of  $\mathcal{S}'(\mathbb{R}^3)$ ,  $(\pi\|x\|^2)^{-1}$  [15, Chap 2, Sect 3.3], the  $p$ -mechanisation (see equation (6)) of  $H$  is

$$B_H(s, x, y) = \left( -\frac{1}{8\pi^2} \delta(s) \delta(x) \delta^{(2)}(y) - \delta(s) \frac{1}{\pi\|x\|^2} \delta(y) \right). \quad (34)$$

This is a distribution in the space  $\mathcal{S}'(\mathbb{H}^3)$ . Three classical constants of the motion are the components of the classical angular momentum vector. The  $i$ th component of the classical angular momentum vector is

$$l_i = \epsilon_{ijk} q_j p_k \quad (35)$$

(in the above equation we have used summation convention). The  $p$ -mechanisation of the  $i$ th component of angular momentum is

$$L_i = -\frac{1}{4\pi^2} \epsilon_{ijk} \delta(s) \delta_{(j)}^{(1)}(x) \delta_{(k)}^{(1)}(y). \quad (36)$$

$\delta_{(j)}^{(1)}(x)$  represents the distribution  $\frac{\partial}{\partial x_j} \delta(x_1, x_2, x_3)$ . Three more constants of the classical motion are the three components of the classical Laplace–Runge–Lenz vector. The  $i$ th component of the Laplace–Runge–Lenz vector can be written as

$$f_i = \epsilon_{ijk} l_j p_k + \frac{q_i}{r}. \quad (37)$$

The  $p$ -mechanisation of this observable is

$$F_i = \frac{1}{2\pi i} \epsilon_{ijk} L_j * \delta(s) \delta(x) \delta_{(k)}^{(1)}(y) + \frac{1}{2\pi i} \delta(s) \delta_{(i)}^{(1)}(x) \delta(y) * \delta(s) \frac{1}{\pi\|x\|^2} \delta(y). \quad (38)$$

The Hamiltonian along with both the angular momentum vector and the Lenz vector are shown to satisfy an  $o(4)$  symmetry [14] under both the Poisson brackets and the quantum commutator. Using the commutation of the left and right

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<sup>3)</sup> Here we have taken all constants equal to one to reduce the technicalities in the calculations.

invariant vector fields [3] along with the results

$$\sum_{j=1}^3 \frac{\partial}{\partial x_j} x_j \frac{1}{\|x\|^2} = \frac{1}{\|x\|^2},$$

$$x_i \frac{\partial}{\partial x_j} \frac{1}{\|x\|^2} = -\frac{2x_i x_j}{\|x\|^4} = x_j \frac{\partial}{\partial x_i} \frac{1}{\|x\|^2},$$

we get the same  $o(4)$  symmetry under the universal brackets (see equation (8)). This means that if  $\xi$  and  $\eta$  are elements of  $\mathbb{R}^3$  then

$$\{[L.\xi, L.\eta]\} = L.(\xi \times \eta), \quad (39)$$

$$\{[L.\xi, F.\eta]\} = F.(\xi \times \eta), \quad (40)$$

$$\{[F.\xi, F.\eta]\} = -2H * L.(\xi \times \eta). \quad (41)$$

The Kepler/Coulomb  $p$ -dynamic equation (see equation (9)) for an arbitrary  $p$ -mechanical observable,  $B$ , takes the form:

$$\frac{dB}{dt} = -\sum_{j=1}^3 y_j \frac{\partial B}{\partial x_j} + \frac{1}{\pi} \int_{\mathbb{R}^3} \frac{1}{\|x-x'\|^2} [B(s + \frac{1}{2}y(x-x'), x', y) - B(s + \frac{1}{2}y(x'-x), x', y)] dx'.$$

This equation is very hard to analyse due to being the mixture of a differential equation and an integral equation. This shows us that taking this approach to obtain relations between classical and quantum mechanics is not suitable for this system.

## 6.2 The Klauder coherent states for the hydrogen atom

Ever since Schrödinger introduced the harmonic oscillator coherent states the hunt has been on to find a set of states which have the same properties for the hydrogen atom. Many efforts have been made which possess some of the properties of the harmonic oscillator coherent states, but finding a set of states which possessed all the same properties for the hydrogen atom was never achieved. One of the best attempts was done by Klauder in his ground breaking paper [16] — a set of coherent states for the hydrogen atom were introduced which had the properties of being: continuous in their label, temporally stable and satisfying a resolution of unity for the bound state portion of the hydrogen atom. Unfortunately they are not minimal uncertainty states, but for our purposes we do not require this property. We now give a brief overview of these coherent states.

Before we can define the Kepler/Coulomb coherent states we need to introduce the angular-momentum coherent states adapted to the Kepler/Coulomb problem

[16, Eq. 15]

$$\begin{aligned} \psi_{(n,\bar{\Omega})}(r, \theta, \phi) &= \sum_{l=0}^n \sum_{m=-l}^l \left[ \frac{(2l)!}{(l+m)!(l-m)!} \right]^{1/2} \sin^{l-m} \left( \frac{\bar{\theta}}{2} \right) \cdot \cos^{l+m} \left( \frac{\bar{\theta}}{2} \right) \times \\ &\quad \times e^{-i(m\bar{\phi}+l\bar{\psi})} \psi_{(n+1,l,m)}(r, \theta, \phi) (2l+1)^{1/2}. \end{aligned}$$

It is important to note that for labelling the coherent states a bar is used over  $\bar{\Omega} = (\bar{\theta}, \bar{\phi}, \bar{\psi})$ , to show that they are different from the  $\theta$  and  $\phi$  in the domain of the function. The functions  $\psi_{(n,l,m)}(r, \theta, \phi)$  are the bound state (negative energy) eigenfunctions for the Kepler/Coulomb Hamiltonian [17]. We denote by  $\mathbb{BS}$  the space spanned by the vectors  $\psi_{(n,l,m)}$ .

We let  $\mathcal{AM}_n$  denote the  $n$ th angular momentum subspace — that is the space spanned by the angular momentum eigenfunctions  $\psi_{(l,m)}$  [17], for  $0 \leq l \leq n$  and  $-l \leq m \leq l$ . It is shown in [16] that these coherent states satisfy a resolution of the identity in the subspace  $\mathcal{AM}_n$ , that is

$$\int \langle \psi, \psi_{(n,\bar{\Omega})} \rangle \psi_{(n,\bar{\Omega})} \sin(\bar{\theta}) d\bar{\theta} d\bar{\phi} d\bar{\psi} = \begin{cases} \psi, & \text{if } \psi \in \mathcal{AM}_n; \\ 0, & \text{otherwise.} \end{cases}$$

Now we can define the Kepler/Coulomb coherent states as<sup>4</sup>

$$\psi_{(\sigma,\gamma,\bar{\Omega})} = e^{-\sigma^2} \sum_{n=0}^{\infty} \left( \frac{\sigma^n \exp[-2\pi\gamma \cdot (ih(n+1)^2)^{-1}]}{(n!)^{1/2}} \right) \psi_{(n,\bar{\Omega})}. \quad (42)$$

For later use we define the measure  $\nu(\sigma, \gamma, \bar{\Omega})$  as

$$\begin{aligned} \int f(\sigma, \gamma, \bar{\Omega}) d\nu(\sigma, \gamma, \bar{\Omega}) &= \\ &= \int_0^\pi \int_0^{2\pi} \int_0^{2\pi} \lim_{\Theta \rightarrow \infty} \frac{1}{2\Theta} \int_{-\Theta}^{\Theta} \int_0^\infty f(\sigma, \gamma, \bar{\Omega}) \sin(\bar{\theta}) d\sigma d\gamma d\bar{\theta} d\bar{\phi} d\bar{\psi}. \end{aligned} \quad (43)$$

We also define the measure  $\mu(r, \theta, \phi)$  by

$$\int \psi(r, \theta, \phi) d\mu(r, \theta, \phi) = \int_0^{2\pi} \int_0^\pi \int_0^\infty \psi(r, \theta, \phi) r^2 \sin(\theta) dr d\theta d\phi.$$

One property of the coherent states defined in equation (42) is that they satisfy a resolution of the identity for the bound states of the Kepler/Coulomb Hamiltonian [16, Eq. 18], that is

$$\int \langle \psi, \psi_{(\sigma,\gamma,\bar{\Omega})} \rangle \psi_{(\sigma,\gamma,\bar{\Omega})} d\nu(\sigma, \gamma, \bar{\Omega}) = \begin{cases} \psi, & \text{if } \psi \in \mathbb{BS}; \\ 0, & \text{otherwise.} \end{cases}$$

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<sup>4</sup>These are by no means the unique choice of Kepler/Coulomb coherent states. The weights  $e^{-\sigma^2}$  and  $n!$  may be changed as described in [16]. In various papers [22, 23, 25, 26] various suggestions on other choices of these weights are given.

Another property of the coherent states is

$$\rho_h(B_H)\psi_{(\sigma,\gamma,\bar{\Omega})} = e^{-\sigma^2} \sum_{n=0}^{\infty} \left( -\frac{\omega\sigma^n \exp\left[2\pi\gamma \cdot (i\hbar(n+1)^2)^{-1}\right]}{(n+1)^2(n!)^{1/2}} \right) \psi_{(n,\bar{\Omega})}. \quad (44)$$

This can also be realized as

$$-\frac{2\pi}{i\hbar} \rho_h(B_H)\psi_{(\sigma,\gamma,\bar{\Omega})} = \omega \frac{\partial}{\partial \gamma} \psi_{(\sigma,\gamma,\bar{\Omega})}(r, \theta, \phi). \quad (45)$$

The above two equations are alternative realizations of the temporal stability [16] property of the Kepler/Coulomb states.

### 6.3 A Hilbert space for the Kepler/Coulomb problem

In this section we introduce a new Hilbert space which is suitable for modelling the quantum mechanical Kepler/Coulomb problem. Two models of quantum mechanics are said to be equivalent if all the transition amplitudes are the same [19]. We show in this section that for a subset of states and a subset of observables a model using this new Hilbert space will be equivalent to the standard model (that is, the model using the irreducible unitary Schrödinger representation on  $L^2(\mathbb{R}^3)$ ). Initially we define a new space.

**Definition 6.1.** *We define the Kepler/Coulomb space, which we denote  $\mathbb{K}\mathbb{C}$ , to be*

$$\mathbb{K}\mathbb{C} = \left\{ f(\sigma, \gamma, \bar{\Omega}) = \int \psi(r, \theta, \phi) \overline{\psi_{(\sigma,\gamma,\bar{\Omega})}(r, \theta, \phi)} d\mu(r, \theta, \phi) : \psi \in \mathbb{B}\mathbb{S} \right\}. \quad (46)$$

The inner product of  $f_1, f_2 \in \mathbb{K}\mathbb{C}$  is given by

$$\langle f_1, f_2 \rangle = \int f_1(\sigma, \gamma, \bar{\Omega}) \overline{f_2(\sigma, \gamma, \bar{\Omega})} d\nu(\sigma, \gamma, \bar{\Omega}),$$

where  $\nu$  is the measure defined in equation (43). We can take the completion of this space with respect to this inner product to obtain a Hilbert space. We have a map  $\mathcal{K}_1 : \mathbb{B}\mathbb{S} \rightarrow \mathbb{K}\mathbb{C}$  given by

$$\begin{aligned} (\mathcal{K}_1(\psi))(\sigma, \gamma, \bar{\Omega}) &= \int \psi(r, \theta, \phi) \overline{\psi_{(\sigma,\gamma,\bar{\Omega})}(r, \theta, \phi)} d\mu(r, \theta, \phi) = \\ &= \langle \psi, \psi_{(\sigma,\gamma,\bar{\Omega})} \rangle_{\mathcal{F}(\mathbb{S}\mathbb{P}^3)}. \end{aligned} \quad (47)$$

We use the notation  $f_{(r,\theta,\phi)}$  to denote  $\psi_{(\sigma,\gamma,\bar{\Omega})}$  as an element of  $\mathbb{K}\mathbb{C}$ .

**Lemma 6.2.**  $\mathcal{K}_1$  is a unitary operator and has inverse  $\mathcal{K}_1^{-1} : \mathbb{K}\mathbb{C} \rightarrow \mathbb{B}\mathbb{S}$

$$\mathcal{K}_1^{-1}f = \int f(\sigma, \gamma, \bar{\Omega}) \psi_{(\sigma,\gamma,\bar{\Omega})}(r, \theta, \phi) d\nu(\sigma, \gamma, \bar{\Omega}). \quad (48)$$

*Proof.* Both of these assertions follow from the fact that the coherent states  $\psi_{(\sigma,\gamma,\bar{\Omega})}$  satisfy a resolution of the identity for the bound states of the Kepler/Coulomb problem.  $\square$

**Theorem 6.3.** *If  $A$  is an operator on  $\mathbb{BS}$  and  $\psi_1, \psi_2 \in \mathbb{BS}$ , then if we let  $\tilde{A} = \mathcal{K}_1 A \mathcal{K}_1^{-1}$ ,  $f_1 = \mathcal{K}_1 \psi_1$  and  $f_2 = \mathcal{K}_1 \psi_2$  we have*

$$\langle A\psi_1, \psi_2 \rangle = \langle \tilde{A}f_1, f_2 \rangle.$$

*Proof.* Using Lemma 6.2 we have

$$\begin{aligned} \langle \tilde{A}f_1, f_2 \rangle &= \langle \mathcal{K}_1 A \mathcal{K}_1^{-1} \mathcal{K}_1 \psi_1, \mathcal{K}_1 \psi_2 \rangle = \\ &= \langle \mathcal{K}_1 A \psi_1, \mathcal{K}_1 \psi_2 \rangle = \\ &= \langle A\psi_1, \psi_2 \rangle. \end{aligned}$$

$\square$

This theorem means that if we transform the usual model of quantum mechanics by the operator  $\mathcal{K}_1$  then our new model is equivalent for operators which preserve  $\mathbb{BS}$  and states which are in  $\mathbb{BS}$ . So this new Hilbert space is suitable for modelling quantum mechanics as long as we are only considering operators which preserve  $\mathbb{BS}$  and states which are bound states for the Kepler/Coulomb problem. Unfortunately this model does not extend to all observables and so we cannot obtain a representation of the Heisenberg group on this space. We now show that for the Kepler/Coulomb problem the time evolution in our new Hilbert space,  $\mathbb{KC}$  is just a shift in the  $\gamma$  variable.

**Theorem 6.4.** *If  $\hat{H}$  is the operator on  $\mathbb{BS}$  equal to  $\rho_h(B_H)$  then*

$$\tilde{H} = \mathcal{K}_1 \hat{H} \mathcal{K}_1^{-1} = \frac{i\hbar}{2\pi} \omega \frac{\partial}{\partial \gamma}.$$

*Proof.* It is clear that  $\hat{H}$  will preserve the space  $\mathbb{BS}$  and so is an operator on this space. If we let  $f$  be an arbitrary element of  $\mathbb{KC}$  then  $f = \mathcal{K}_1 \psi$  for some  $\psi \in \mathbb{BS}$

$$\begin{aligned} \tilde{H}f &= \tilde{H} \mathcal{K}_1 \psi = \\ &= \mathcal{K}_1 \hat{H} \psi = \\ &= \langle \hat{H} \psi, \psi_{(\sigma,\gamma,\bar{\Omega})} \rangle = \\ &= \langle \psi, \hat{H} \psi_{(\sigma,\gamma,\bar{\Omega})} \rangle = \end{aligned} \tag{49}$$

$$\begin{aligned} &= \left\langle \psi, -\frac{i\hbar}{2\pi} \omega \frac{\partial}{\partial \gamma} \psi_{(\sigma,\gamma,\bar{\Omega})} \right\rangle = \tag{50} \\ &= \frac{i\hbar}{2\pi} \omega \frac{\partial}{\partial \gamma} \langle \psi, \psi_{(\sigma,\gamma,\bar{\Omega})} \rangle = \\ &= \frac{i\hbar}{2\pi} \omega \frac{\partial f}{\partial \gamma}. \end{aligned}$$

At (49) we have used the fact that  $\hat{H}$  is a self adjoint operator and at (50) we have used equation (45).  $\square$

The Schrödinger equation in  $\mathbb{K}\mathbb{C}$  is

$$\frac{df}{dt} = \frac{2\pi}{i\hbar} \tilde{H}f = \omega \frac{\partial f}{\partial \gamma}.$$

So the time evolution of an arbitrary  $f(t; \sigma, \gamma, \bar{\Omega}) \in \mathbb{K}\mathbb{C}$  is given by

$$f(t; \sigma, \gamma, \bar{\Omega}) = f_0(\sigma, \gamma + \omega t, \bar{\Omega}),$$

where  $f_0(\sigma, \gamma, \bar{\Omega}) = f(0; \sigma, \gamma, \bar{\Omega})$  the initial value of the state at time  $t = 0$ . The eigenfunctions in  $\mathbb{K}\mathbb{C}$  of the operator  $\tilde{H} = \frac{i\hbar}{2\pi} \omega \frac{\partial}{\partial \gamma}$  are

$$\begin{aligned} f_{(n,l,m)}(\sigma, \gamma, \bar{\Omega}) &= e^{-\sigma^2} \frac{\sigma^n \exp \left[ -2\pi\gamma \cdot (i\hbar n^2)^{-1} \right]}{(n!)^{1/2}} \left[ \frac{(2l)!}{(l+m)!(l-m)!} \right]^{1/2} \times \\ &\times \sin^{l-m} \left( \frac{\bar{\theta}}{2} \right) \cdot \cos^{l+m} \left( \frac{\bar{\theta}}{2} \right) \cdot e^{-i(m\bar{\phi} + l\bar{\psi})} (2l+1)^{1/2}, \end{aligned}$$

where  $n \in \mathbb{N}$ ,  $l \in \mathbb{N}$  such that  $0 \leq l \leq n$  and  $m \in \mathbb{Z}$  such that  $-l \leq m \leq l$ . These eigenfunctions will have eigenvalue  $-\omega n^{-2}$  with degeneracy  $n^2$ . This agrees with the usual quantum mechanical theory. It is important to note that this model is only suitable for calculating probability amplitudes for states which are in  $\mathbb{B}\mathbb{S}$  and observables which preserve  $\mathbb{B}\mathbb{S}$ . However we will describe in Subsection 6.4 how this can be extended to model a larger set of states.

#### 6.4 Generalizations

We now show how the above approach for the Kepler/Coulomb problem can be extended to any quantum mechanical system with a discrete spectrum. Furthermore we show that this approach can be extended to include systems with discrete and continuous spectra. This is all done by facilitating the extensions of Klauder's coherent states.

Since Klauder discovered his coherent states for the hydrogen atom there have been many extensions. Majumdar and Sharatchandra have written a paper [20] discussing relations between coherent states for the hydrogen atom and the action angle variables for the Kepler problem. Fox [22] extended this approach to show how these states could be realized as Gaussians. Crawford [23] described an extension which could model general systems with energy degeneracies. This work used the Perelomov coherent states [25] for the degeneracy group. Since these coherent states satisfy both a resolution of unity for the set of states in question and are temporally stable, the associated Hilbert spaces can be obtained in exactly the same way as in Section 6.3. The proofs will almost follow word for word.

If the set of eigenfunctions for the Hamiltonian in question spans the entire space then  $\mathcal{K}_1$  from equation (47) will be unitary, bijective, invertible and defined on the whole of  $\mathbb{K}\mathbb{C}$ . This means that the Hilbert space we obtain will be able to deal with any observable and any state. Furthermore  $\mathcal{K}_1\rho_h\mathcal{K}_1^{-1}$  will be a unitary irreducible representation of the Heisenberg group which is unitarily equivalent to the Schrödinger representation. This representation would be able to model probability amplitudes for any quantum mechanical state and quantum mechanical observable.

We can also extend our approach to systems with both discrete and continuous spectra. The extension of the original coherent states to systems with both discrete and continuous spectra is given in [25, 26]. Since these coherent states satisfy a resolution of the identity and are temporally stable we can obtain another Hilbert space by following the proofs in Section 6.3 word for word.

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