On the self-similarities of the rhombic Penrose tilings

Nicolae Cotfas

Faculty of Physics, University of Bucharest
PO Box 76 – 54, Postal Office 76, Bucharest, Romania
E-mail: ncotfas@yahoo.com

We prove that the original Penrose tilings of the plane admit an infinite number of independent scaling factors and an infinite number of inflation centers. Our results are based on the definition of these tilings in terms of strip projection method proposed by Katz and Duneau shortly after the discovery of quasicrystals.

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1 Introduction

The Fibonacci tiling of the line and the Penrose tilings of the plane are among the lower dimensional quasiperiodic tilings used in the theoretical investigation of the possible structures and physical properties of quasicrystals. Katz and Duneau have shown [2] that the tilings of the plane by two rhombus obtained by Penrose by using an inflation and a deflation procedure can be defined in terms of the strip projection method. We use this definition in order to obtain some results concerning the self-similarities of these remarkable tilings.

The method we use in the study of self-similarities is a direct generalization of the method used by Katz and Duneau [2] in the case of an icosahedral tiling. This method has been used up to now only in the case when the representation of the symmetry group in superspace has only two $\mathbb{R}$–irreducible components (corresponding to the orthogonal projectors π and π'). In this case, one looks for scaling factors in the set of the numbers λ satisfying the condition: there is λ' such that the matrix $\lambda\pi + \lambda'\pi'$ has integer entries. The additional condition one has to impose depends on the shape of the window. If the window is a convex set symmetric with respect to the origin then it is sufficient to ask $|\lambda'| < 1$.

In the case of Penrose tilings and in many other cases the representation of the symmetry group in superspace has more than two $\mathbb{R}$–irreducible components. In the cases of interest for quasicrystal physics, the entries of the projectors belong to some quadratic fields and it is possible [1] to decompose the superspace into three invariant subspaces $\mathbf{E}$ ('physical space'), $\mathbf{E}'$ and $\mathbf{E}''$ such that:

1) the representations subduced to $\mathbf{E}$ and $\mathbf{E}'$ are $\mathbb{R}$–irreducible;
2) the projector corresponding to $\mathbf{E}''$ has rational entries.

In this case, we look for scaling factors in the set of the numbers λ satisfying the condition: there exist λ' and λ'' such that $\lambda\pi + \lambda'\pi' + \lambda''\pi''$ has integer entries.

The self-similarities of the set of all the atomic positions are expected to play an important role in the description of the physical properties of quasicrystals. Our aim is to present some results concerning the self-similarities of rhombic Penrose tilings, but it is not our purpose to determine all the self-similarities.

2 Penrose tilings

In this section we resume the definition of the rhombic Penrose tilings in terms of the strip projection method, definition proposed by Katz and Duneau [2]. The relation

$$g(x_1,x_2,x_3,x_4,x_5)=(x_5,x_1,x_2,x_3,x_4) \tag{1}$$

defines a linear representation of the cyclic group $C_5=\langle g\mid g^5=e\rangle=\{e,g,g^2,g^3,g^4\}$ in the five–dimensional Euclidean space $\mathbb{E}_5=(\mathbb{R}^5,\langle,\rangle)$, where

$$\langle(x_1,x_2,x_3,x_4,x_5),(y_1,y_2,y_3,y_4,y_5)\rangle=\sum_{i=1}^{5}x_iy_i\,,\quad \|x\|=\sqrt{\langle x,x\rangle}\,. \tag{2}$$

This representation is a sum of three $\mathbb{R}$–irreducible representations. The corresponding invariant orthogonal subspaces are

$$\begin{aligned}\mathbf{E}&=\{\alpha v_1+\beta v_2\mid \alpha,\beta\in\mathbb{R}\}\,,\\ \mathbf{E}'&=\{\alpha v_1'+\beta v_2'\mid \alpha,\beta\in\mathbb{R}\}\,,\\ \mathbf{E}''&=\{\alpha w\mid \alpha\in\mathbb{R}\}\,,\end{aligned} \tag{3}$$

where

$$\begin{aligned}v_1&=\left(1,\cos\tfrac{2\pi}{5},-\cos\tfrac{\pi}{5},-\cos\tfrac{\pi}{5},\cos\tfrac{2\pi}{5}\right),\\ v_2&=\left(0,\sin\tfrac{2\pi}{5},\sin\tfrac{\pi}{5},-\sin\tfrac{\pi}{5},-\sin\tfrac{2\pi}{5}\right),\\ v_1'&=\left(1,-\cos\tfrac{\pi}{5},\cos\tfrac{2\pi}{5},\cos\tfrac{2\pi}{5},-\cos\tfrac{\pi}{5}\right),\\ v_2'&=\left(0,\sin\tfrac{\pi}{5},-\sin\tfrac{2\pi}{5},\sin\tfrac{2\pi}{5},-\sin\tfrac{\pi}{5}\right),\\ w&=\left(\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5}\right).\end{aligned} \tag{4}$$

The vectors $\varepsilon_1=(1,0,0,0,0)$, $\varepsilon_2=(0,1,0,0,0)$,..., $\varepsilon_5=(0,0,0,0,1)$ form the canonical basis of $\mathbb{E}_5$. The matrices of the orthogonal projectors π, π' and π'' corresponding to $\mathbf{E}$, $\mathbf{E}'$ and $\mathbf{E}''$ in this basis are

$$\pi=\mathcal{M}(\tfrac{2}{5},-\tfrac{\tau'}{5},-\tfrac{\tau}{5})\,,\quad \pi'=\mathcal{M}(\tfrac{2}{5},-\tfrac{\tau}{5},-\tfrac{\tau'}{5})\,,\quad \pi''=\mathcal{M}(\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5})\,, \tag{5}$$

where $\tau=\frac{1+\sqrt{5}}{2}$, $\tau'=\frac{1-\sqrt{5}}{2}$ and

$$\mathcal{M}(\alpha,\beta,\gamma)=\begin{pmatrix}\alpha&\beta&\gamma&\gamma&\beta\\ \beta&\alpha&\beta&\gamma&\gamma\\ \gamma&\beta&\alpha&\beta&\gamma\\ \gamma&\gamma&\beta&\alpha&\beta\\ \beta&\gamma&\gamma&\beta&\alpha\end{pmatrix}. \tag{6}$$

The projection $\pi(\mathcal{L})$ (resp. $\pi'(\mathcal{L})$) of the five–dimensional lattice $\mathcal{L}=\mathbb{Z}^5$ is a $\mathbb{Z}$–module dense in E (resp. E') generated by the vectors $\pi\varepsilon_1$, $\pi\varepsilon_2$, ..., $\pi\varepsilon_5$ (resp.

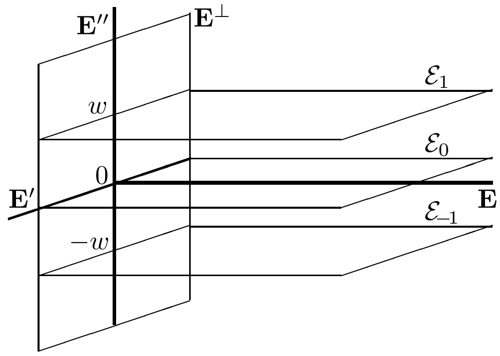

Fig. 1. The decompositions $\mathbb{E}_5 = \mathbf{E} \oplus \mathbf{E}^\perp = \mathbf{E} \oplus \mathbf{E}' \oplus \mathbf{E}''$ and the subspaces $\mathcal{E}_n$.

$\pi'\varepsilon_1,\ \pi'\varepsilon_2,\ \ldots,\ \pi'\varepsilon_5)$ which point to the vertices of a regular pentagon of center $(0,0,0,0,0)$. The mapping $\pi^\perp = \pi' + \pi''$ is the orthogonal projector corresponding to the subspace $\mathbf{E}^\perp = \mathbf{E}' \oplus \mathbf{E}''$. Since

$$\pi''(\mathcal{L}) = \mathbb{Z}w = \{nw \mid n \in \mathbb{Z}\}\,, \tag{7}$$

the lattice $\mathcal{L}$ is contained in the union of parallel and equidistant subspaces $\bigcup_{n\in\mathbb{Z}} \mathcal{E}_n$, where (see figure 1)

$$\mathcal{E}_n = nw + \mathbf{E} \oplus \mathbf{E}' \tag{8}$$

and the $\mathbb{Z}$–module $\pi^\perp(\mathcal{L})$ is contained in the union of parallel and equidistant planes $\mathbb{Z}w + \mathbf{E}' = \bigcup_{n\in\mathbb{Z}}(nw + \mathbf{E}')$.

Let $\mathcal{L}_n = \mathcal{L} \cap \mathcal{E}_n$, Ω be the set of all the points lying in the interior of the regular pentagon with vertices $\pi'\varepsilon_1,\ \pi'\varepsilon_2,\ \ldots,\ \pi'\varepsilon_5$, and let

$$\mathbf{W} = t + \{(x_1, x_2, x_3, x_4, x_5) \mid 0 < x_i < 1\}\,, \tag{9}$$

where the translation vector $t \in \mathbf{E}'$ is chosen such that the boundary $\partial\mathcal{W}$ of the set $\mathcal{W} = \pi^\perp(\mathbf{W})$ does not contain any element of $\pi^\perp(\mathcal{L})$. This is possible since the set $\pi^\perp(\mathcal{L}) + \partial\mathcal{W}$ has Lebesgue measure 0. The set defined in terms of the strip projection method

$$\mathcal{P} = \{\pi x \mid x \in \mathcal{L},\ \pi^\perp x \in \mathcal{W}\} \tag{10}$$

is the *set of all the vertices of a rhombic Penrose tiling* [2].

The set $\mathbf{W} \cap \mathcal{E}_n$ is non-empty only for $n \in \{1,2,3,4\}$, and

$$\begin{aligned}
\pi^\perp(\mathbf{W} \cap \mathcal{E}_1) &= \pi'(\mathbf{W} \cap \mathcal{E}_1) + w = t + \Omega + w\,,\\
\pi^\perp(\mathbf{W} \cap \mathcal{E}_2) &= \pi'(\mathbf{W} \cap \mathcal{E}_2) + 2w = t - \tau\Omega + 2w\,,\\
\pi^\perp(\mathbf{W} \cap \mathcal{E}_3) &= \pi'(\mathbf{W} \cap \mathcal{E}_3) + 3w = t + \tau\Omega + 3w\,,\\
\pi^\perp(\mathbf{W} \cap \mathcal{E}_4) &= \pi'(\mathbf{W} \cap \mathcal{E}_4) + 4w = t - \Omega + 4w\,.
\end{aligned} \tag{11}$$

Therefore, $\mathcal{P}$ is a union of four sets

$$\mathcal{P}=\bigcup_{n=1}^{4}\left\{\pi x \;\middle|\; x\in\mathcal{L}_n,\ \pi^{\perp}x\in\pi^{\perp}(\mathbf{W}\cap\mathcal{E}_n)\right\}. \tag{12}$$

The translation along $\mathbf{E}''$

$$T:\mathbb{E}_5\longrightarrow\mathbb{E}_5 : x\mapsto x+5w \tag{13}$$

is bijective, $\pi x=\pi(Tx)$, $\pi' x=\pi'(Tx)$, $T(\mathcal{E}_n)=\mathcal{E}_{n+5}$ and $T(\mathcal{L}_n)=\mathcal{L}_{n+5}$. Denoting $\mathcal{W}_{5k}=\emptyset$, $\mathcal{W}_{5k+1}=t+\Omega$, $\mathcal{W}_{5k+2}=t-\tau\Omega$, $\mathcal{W}_{5k+3}=t+\tau\Omega$ and $\mathcal{W}_{5k+4}=t-\Omega$ for any $k\in\mathbb{Z}$, we can [2] re-write the definition of $\mathcal{P}$ as

$$\mathcal{P}=\bigcup_{n\in\mathbb{Z}}\left\{\pi x \;\middle|\; x\in\mathcal{L}_n,\ \pi' x\in\mathcal{W}_n\right\}. \tag{14}$$

3 Self-similarities of Penrose tilings

The number λ is called a *scaling factor* of $\mathcal{P}$ if there is $y\in\mathbf{E}$ such that $\mathcal{P}$ is invariant under the affine similarity [3]

$$\Lambda:\mathbf{E}\longrightarrow\mathbf{E}\,,\quad \Lambda x=y+\lambda(x-y)\,, \tag{15}$$

that is, if $\Lambda(\mathcal{P})\subset\mathcal{P}$. In this case we say that y is an *inflation center* corresponding to λ, and Λ is a *self-similarity* of $\mathcal{P}$. Since Λ maps each segment of straight line into a segment of straight line, any self-similarity of the set $\mathcal{P}$ is at the same time a self-similarity of the corresponding rhombic tiling. The set $\Lambda(\mathcal{P})$ is the set of all the vertices of a similar tiling, inflated by λ.

The subspaces $\mathbf{E}$, $\mathbf{E}'$ and $\mathbf{E}''$ are invariant under the linear transformation

$$S:\mathbb{E}_5\longrightarrow\mathbb{E}_5\,,\quad S=\lambda\pi+\lambda'\pi'+\lambda''\pi'' \tag{16}$$

for any λ, λ', $\lambda''\in\mathbb{R}$. If the matrix of S in the basis $\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_5\}$ has integer entries then $S(\mathcal{L})\subset\mathcal{L}$.

Lemma.
a) If S has integer entries then $\lambda''\in\mathbb{Z}$.
b) If j, k and l are integer numbers then

$$S=(j+k\tau)\pi+(j+k\tau')\pi'+l\pi'' \tag{17}$$

has integer entries if and only if $2k-j+l\in 5\mathbb{Z}$.

Proof. a) From $S=\mathcal{M}(p,q,r)$ we get $\lambda''=p+2q+2r$.
b) We have

$$S=\mathcal{M}\left(\frac{2k-j+l}{5}+k,\frac{2k-j+l}{5},\frac{2k-j+l}{5}-k\right). \qquad \blacksquare$$

Theorem. *If $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ are such that*

$$2k - j + 1 \in 5\mathbb{Z} \quad \text{and} \quad \lambda' = j + k\tau' \in \left(-\frac{1+\sqrt{5}}{4}, 1 \right), \tag{18}$$

then $\lambda = j + k\tau$ is a scaling factor of $\mathcal{P}$.

Proof. Since $2k - j + 1 \in 5\mathbb{Z}$ the matrix corresponding to

$$S = (j + k\tau)\pi + (j + k\tau')\pi' + \pi'' \tag{19}$$

has integer entries, and $S(\mathcal{L}_n) \subset \mathcal{L}_n$. From $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ and (18) it follows that $\lambda'\Omega \subset \Omega$, whence $\lambda'(\mathcal{W}_n - t) + t \subset \mathcal{W}_n$, for any $n \in \mathbb{Z}$. More than that, there is $\delta > 0$ such that $\lambda'(\mathcal{W}_n - u') + u' \subset \mathcal{W}_n$, for any $n \in \mathbb{Z}$ and for any $u' \in \mathbf{E}'$ with $||u' - t|| < \delta$. Since $\pi'(\mathcal{L}_0)$ is dense in $\mathbf{E}'$, the set

$$\mathcal{Q}_\lambda = \{u \mid u \in \mathcal{L}_0,\ \|\pi' u - t\| < \delta\} \tag{20}$$

is an infinite set. Let $u \in \mathcal{Q}_\lambda$ be a fixed element, and let

$$A : \mathbb{E}_5 \longrightarrow \mathbb{E}_5\,, \quad Ax = u + S(x - u)\,. \tag{21}$$

Since

$$\left.\begin{array}{l} x \in \mathcal{L}_n \\ \pi' x \in \mathcal{W}_n \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} Ax \in \mathcal{L}_n \\ \pi'(Ax) \in \mathcal{W}_n \end{array}\right. \tag{22}$$

from the definition of $\mathcal{P}$ it follows that

$$\pi x \in \mathcal{P} \implies \pi(Ax) = \pi u + \lambda(\pi x - \pi u) \in \mathcal{P}\,, \tag{23}$$

that is, λ is a scaling factor of $\mathcal{P}$ and πu a corresponding inflation center. ■

Denoting $2k - j + 1 = 5m$ we get $\lambda' = 1 - 5m + k(2 + \tau')$. The condition (18) is satisfied by an infinite number of pairs $(j, k) \in \mathbb{Z}^2$. Therefore, any original Penrose tiling [2] admits an infinite number of scaling factors, and for each of them there is an infinite number of inflation centers. The self-similarities determined above are not all the self-similarities of $\mathcal{P}$. For example, one can prove [2] that $\lambda = -\tau$ is a scaling factor of $\mathcal{P}$ by using the linear transformation $S = -\tau\pi - \tau'\pi' + 2\pi''$ and the relation (14).

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