Characters of $D = 4$ conformal supersymmetry

V.K. Dobrev

Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

We give character formulae for the positive energy unitary irreducible representations of the N–extended $D = 4$ conformal superalgebras $su(2, 2/N)$. Using these we also derive decompositions of long superfields as they descend to the unitarity threshold. These results are also applicable to irreps of the complex Lie superalgebras $sl(4/N)$.

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1 Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, cf. extensive bibliography in [1]. This makes the classification of the UIRs of the conformal superalgebras very important. Until recently such classification was known only for the $D = 4$ superconformal algebras $su(2, 2/1)$ [2] and $su(2, 2/N)$ (for arbitrary N) [3]. Recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [4] (some results being conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [5]. Finally, the cases $D = 9$, 10, 11 were treated by finding the UIRs of $osp(1/2n)$, [6].

Once we know the UIRs of a (super-)algebra the next question is to find their characters, since these give the spectrum which is important for the applications. This is the question we address in this paper for the UIRs of $D = 4$ conformal superalgebras $su(2, 2/N)$. From the mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra $sl(4/N)$. But for $su(2, 2/N)$ even the UIRs above the unitarity threshold are truncated for small values of spin and isospin.

Thus, we need detailed knowledge about the structure of the UIRs from the representation–theoretical point of view. Fortunately, such information is contained in [3, 7–9]. The present paper is a more compact version of [1] to which we refer for more extended introduction.

2 Representations of $D = 4$ conformal supersymmetry

2.1 The setting

The conformal superalgebras in $D = 4$ are $\mathcal{G} = su(2, 2/N)$. The even subalgebra of $\mathcal{G}$ is the algebra $\mathcal{G}_0 = su(2, 2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\, ;\, j_1\, ,\, j_2\, ;\, z\, ;\, r_1\, , \ldots , r_{N-1}\,]\,, \tag{1}$$

where d is the conformal weight, j_1, j_2 are non-negative (half–)integers which are Dynkin labels of the finite–dimensional irreps of the $D=4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1+1)(2j_2+1)$, z represents the $u(1)$ subalgebra which is central for $\mathcal{G}_0$ (and for $N=4$ is central for $\mathcal{G}$ itself), and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite–dimensional irreps of the internal (or R) symmetry algebra $su(N)$.

We need the standard triangular decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \,, \tag{2}$$

where $\mathcal{G}^{\mathbb{C}} = sl(4/N)$ is the complexification of $\mathcal{G}$, $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots of $\mathcal{G}^{\mathbb{C}}$, resp., and $\mathcal{H}$ denotes the Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+, \Lambda \in \mathcal{H}^*$ is the lowest weight, and v_0 is the lowest weight vector v_0 such that:

$$X v_0 = 0\,, \quad X \in \mathcal{G}^- \,, \qquad H v_0 = \Lambda(H) v_0\,, \quad H \in \mathcal{H}\,. \tag{3}$$

Further, for simplicity we omit the sign $\otimes$.

The lowest weight Λ is characterized by its values on the Cartan subalgebra $\mathcal{H}$. It is in 1–to–1 correspondence with χ and we shall write $\Lambda = \Lambda(\chi)$, or $\chi = \chi(\Lambda)$.

If a Verma module V^Λ is irreducible then it gives the lowest weight irrep L_Λ with the same weight. If a Verma module V^Λ is reducible then it contains a maximal invariant submodule I^Λ and the lowest weight irrep L_Λ with the same weight is given by factorization: $L_\Lambda = V^\Lambda / I^\Lambda$ [10].

Thus, we need first to know which Verma modules are reducible. The reducibility conditions were given by Kac [10]. A lowest weight Verma module V^Λ is reducible only if at least one of the following conditions is true:

$$(\rho - \Lambda, \beta) = m(\beta,\beta)/2\,, \quad \beta \in \Delta^+\,, \quad (\beta,\beta) \neq 0\,, \quad m \in I\!N\,, \tag{4a}$$

$$(\rho - \Lambda, \beta) = 0\,, \quad \beta \in \Delta^+\,, \quad (\beta,\beta) = 0\,, \tag{4b}$$

where Δ^+ is the positive root system of $\mathcal{G}^{\mathbb{C}}$, $\rho \in \mathcal{H}^*$, $\rho = \rho_{\bar 0} - \rho_{\bar 1}$, where $\rho_{\bar 0}$, $\rho_{\bar 1}$ are the half–sums of the even, odd, resp., positive roots, $(\cdot,\cdot)$ is the standard bilinear product in $\mathcal{H}^*$.

If a condition from (4a) is fulfilled then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair $m, \beta : \Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v'_0 of $V^{\Lambda'}$ to the singular vector $v_s^{m,\beta}$ in V^Λ which is determined by:

$$\begin{aligned} X v_s^{m,\beta} &= 0\,, & X &\in \mathcal{G}^- \,, \\ H v_s^{m,\beta} &= \Lambda'(H) v_0\,, & H &\in \mathcal{H}\,, \quad \Lambda' = \Lambda + m\beta\,. \end{aligned} \tag{5}$$

Explicitly, $v_s^{m,\beta}$ is given by an even polynomial in the positive root generators:

$$v_s^{m,\beta} = P^{m,\beta}\, v_0\,, \quad P^{m,\beta} \in U(\mathcal{G}^+)\,. \tag{6}$$

Thus, the submodule of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)P^{m,\beta}v_0$.

If a condition from (4b) is fulfilled then V^Λ contains a submodule I^β obtained from the Verma module $V^{\Lambda'}$ with shifted weight $\Lambda' = \Lambda + \beta$. In this situation V^Λ contains a singular vector v_s^β which fulfils (5) with $m = 1$. Explicitly, v_s^β is given by an odd polynomial in the positive root generators:

$$v_s^\beta = P^\beta v_0 \,, \quad P^\beta \in U(\mathcal{G}^+) \,. \tag{7}$$

Then we have:

$$I^\beta = U(\mathcal{G}^+) P^\beta v_0 \,, \tag{8}$$

which is smaller than $V^{\Lambda'} = U(\mathcal{G}^+)v_0'$ since this polynomial is Grassmannian:

$$\left(P^\beta\right)^2 = 0 \,. \tag{9}$$

To describe this situation we say that $V^{\Lambda'}$ is **oddly embedded** in V^Λ.

Note, however, that the above formulae describe also more general situations when the difference $\Lambda' - \Lambda = \beta$ is not a root, as used in [8], and below.

The weight shifts $\Lambda' = \Lambda + \beta$, when β is an odd root are called **odd reflections** in [8], and for future reference will be denoted as:

$$\hat{s}_\beta \cdot \Lambda \equiv \Lambda + \beta \,, \quad (\beta, \beta) = 0 \,, \quad (\Lambda, \beta) \neq 0 \,. \tag{10}$$

Each such odd reflection generates an infinite discrete abelian group:

$$\tilde{W}_\beta \equiv \{ (\hat{s}_\beta)^n \,|\, n \in \mathbb{Z} \} \,, \quad \ell((\hat{s}_\beta)^n) = n \,, \tag{11}$$

where the unit element is obviously obtained for $n = 0$, and $(\hat{s}_\beta)^{-n}$ is the inverse of $(\hat{s}_\beta)^n$, and for future use we have also defined the length function $\ell(\cdot)$ on the elements of $\tilde{W}_\beta$. This group acts on the weights Λ extending (10):

$$(\hat{s}_\beta)^n \cdot \Lambda = \Lambda + n\beta \,, \quad n \in \mathbb{Z} \,, \quad (\beta, \beta) = 0 \,, \quad (\Lambda, \beta) \neq 0 \,. \tag{12}$$

Further, to be more explicit we need to recall the root system of $\mathcal{G}^{\mathbb{C}}$ — for definiteness — as used in [8]. The positive root system Δ^+ is comprised from α_{ij}, $1 \leq i < j \leq 4 + N$. The even positive root system $\Delta_{\bar{0}}^+$ is comprised from α_{ij}, with $i, j \leq 4$ and $i, j \geq 5$; the odd positive root system $\Delta_{\bar{1}}^+$ is comprised from α_{ij}, with $i \leq 4$, $j \geq 5$. The simple roots are chosen as in (2.4) of [8]:

$$\gamma_1 = \alpha_{12} \,, \;\; \gamma_2 = \alpha_{34} \,, \;\; \gamma_3 = \alpha_{25} \,, \;\; \gamma_4 = \alpha_{4,4+N} \;\; \gamma_k = \alpha_{k,k+1} \,, \;\; 5 \leq k \leq 3+N \,. \tag{13}$$

Thus, the Dynkin diagram is:

$$\underset{1}{\bigcirc} - \underset{3}{\bigotimes} - \underset{5}{\bigcirc} - \cdots - \underset{3+N}{\bigcirc} - \underset{4}{\bigotimes} - \underset{2}{\bigcirc} \tag{14}$$

This is a non-distinguished simple root system with two odd simple roots [11].

Let $\Lambda = \Lambda(\chi)$. The products of Λ with the simple roots are [8]:

$$\begin{aligned}
(\Lambda,\gamma_a) &= -2j_a\,, \qquad a=1,2\,, && (15a)\\
(\Lambda,\gamma_3) &= \tfrac{1}{2}(d+z') + j_1 - m/N + 1\,, && (15b)\\
(\Lambda,\gamma_4) &= \tfrac{1}{2}(d-z') + j_2 - m_1 + m/N + 1\,, && (15c)\\
&\quad z' \equiv z(1-\delta_{N4})\\
(\Lambda,\gamma_j) &= r_{N+4-j}\,, \qquad 5 \le j \le 3+N\,. && (15d)
\end{aligned}$$

In the case of even roots $\beta \in \Delta_{\bar{0}}^+$ there are six roots α_{ij}, $j \le 4$, coming from the $sl(4)$ factor (which is complexification of $su(2,2)$) and $N(N-1)/2$ roots α_{ij}, $5 \le i$, coming form the $sl(N)$ factor (complexification of $su(N)$).

The reducibility conditions w.r.t. to the positive roots of $sl(4)$ coming from (4) (denoting $m \to n_{ij}$ for $\beta \to \alpha_{ij}$) are:

$$\begin{aligned}
n_{12} &= 1 + 2j_1 \equiv n_1\,, && (16a)\\
n_{23} &= 1 - d - j_1 - j_2 \equiv n_2\,, && (16b)\\
n_{34} &= 1 + 2j_2 \equiv n_3\,, && (16c)\\
n_{13} &= 2 - d + j_1 - j_2 \ = \ n_1 + n_2\,, && (16d)\\
n_{24} &= 2 - d - j_1 + j_2 \ = \ n_2 + n_3\,, && (16e)\\
n_{14} &= 3 - d + j_1 + j_2 \ = \ n_1 + n_2 + n_3\,. && (16f)
\end{aligned}$$

Thus, reducibility conditions (16a,c) are fulfilled automatically for $\Lambda(\chi)$ with χ from (1) since we always have: $n_1, n_3 \in I\!N$.

The reducibility conditions w.r.t. to the positive roots of $sl(N)$ are all fulfilled for $\Lambda(\chi)$ with χ from (1). In particular, for the simple roots from those condition (4) is fulfilled with $\beta \to \gamma_j$, $m = 1 + r_{N+4-j}$ for every $j = 5, 6, \ldots, N+3$.

The reducibility conditions for the $4N$ odd positive roots of $\mathcal{G}$ are [7,8]:

$$\begin{aligned}
d &= d^1_{Nk} - z\delta_{N4}\,, && (17a)\\
d^1_{Nk} &\equiv 4 - 2k + 2j_2 + z + 2m_k - 2m/N\,,\\
d &= d^2_{Nk} - z\delta_{N4}\,, && (17b)\\
d^2_{Nk} &\equiv 2 - 2k - 2j_2 + z + 2m_k - 2m/N\,,\\
d &= d^3_{Nk} + z\delta_{N4}\,, && (17c)\\
d^3_{Nk} &\equiv 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N\,,\\
d &= d^4_{Nk} + z\delta_{N4}\,, && (17d)\\
d^4_{Nk} &\equiv 2k - 2N - 2j_1 - z - 2m_k + 2m/N\,,
\end{aligned}$$

where in all four cases of (17) $k = 1, \ldots, N$, $m_N \equiv 0$, and

$$m_k \equiv \sum_{i=k}^{N-1} r_i\,, \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} k r_k\,. \tag{18}$$

We shall consider quotients of Verma modules factoring out the *even* submodules for which the reducibility conditions are always fulfilled. Before this we recall the root vectors following [8]. The positive (negative) root vectors corresponding to α_{ij}, $(-\alpha_{ij})$, are denoted by X^+_{ij}, (X^-_{ij}). In the $su(2,2/N)$ matrix notation the convention of [8], (2.7), is:

$$\begin{aligned} X^+_{ij} &= \begin{cases} e_{ji} & \text{for } (i,j) = (3,4), (3,j), (4,j), \quad 5 \le j \le N+4\,, \\ e_{ij} & \text{otherwise}\,, \end{cases} \\ X^-_{ij} &= {}^t(X^+_{ij})\,, \end{aligned} \tag{19}$$

where e_{ij} are $(N+4)\times(N+4)$ matrices with all elements zero except the element equal to 1 on the intersection of the i–th row and j–th column. The simple root vectors X^+_i follow the notation of the simple roots γ_i (13):

$$X^+_1 \equiv X^+_{12}\,,\ X^+_2 \equiv X^+_{34}\,,\ X^+_3 \equiv X^+_{25}\,,\ X^+_4 \equiv X^+_{4,4+N}\,,\ X^+_k \equiv X^+_{k,k+1}\,,\ 5 \le k \le 3{+}N\,. \tag{20}$$

The mentioned submodules are generated by the singular vectors related to the even simple roots γ_1, γ_2,γ_5, $\ldots$, γ_{N+3} [8]:

$$v^1_s = (X^+_1)^{1+2j_1} v_0\,, \tag{21a}$$
$$v^2_s = (X^+_2)^{1+2j_2} v_0\,, \tag{21b}$$
$$v^j_s = (X^+_j)^{1+r_{N+4-j}} v_0\,, \quad j = 5,\ldots,N+3\,. \tag{21c}$$

The corresponding submodules are $I^\Lambda_k = U(\mathcal{G}^+) v^k_s$, and the invariant submodule to be factored out is:

$$I^\Lambda_c = \bigcup_k I^\Lambda_k \tag{22}$$

Thus, instead of V^Λ we shall consider the factor–modules:

$$\tilde{V}^\Lambda = V^\Lambda / I^\Lambda_c \tag{23}$$

In the factorized modules the singular vectors (21) become null conditions, i.e., denoting by $\widetilde{|\Lambda\rangle}$ the lowest weight vector of $\tilde{V}^\Lambda$, we have:

$$(X^+_1)^{1+2j_1} \widetilde{|\Lambda\rangle} = 0\,, \tag{24a}$$
$$(X^+_2)^{1+2j_2} \widetilde{|\Lambda\rangle} = 0\,, \tag{24b}$$
$$(X^+_j)^{1+r_{N+4-j}} \widetilde{|\Lambda\rangle} = 0\,, \quad j = 5,\ldots,N+3\,. \tag{24c}$$

2.2 Singular vectors and invariant submodules at the unitary reduction points

We first recall the result of [3] (cf. part (i) of the Theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of $su(2,2/N)$:

$$d \ge d_{\max} = \max(d^1_{N1}, d^3_{NN})\,, \tag{25a}$$
$$d = d^4_{NN} \ge d^1_{N1}\,, \quad j_1 = 0\,, \tag{25b}$$
$$d = d^2_{N1} \ge d^3_{NN}\,, \quad j_2 = 0\,, \tag{25c}$$
$$d = d^2_{N1} = d^4_{NN}\,, \quad j_1 = j_2 = 0\,, \tag{25d}$$

where $d_{\max}$ is the threshold of the continuous unitary spectrum. Note that in case (d) we have $d = m_1$, $z = 2m/N - m_1$, and that it is trivial for $N = 1$.

Next we note that if $d > d_{\max}$ the factorized Verma modules are irreducible and coincide with the UIRs L_Λ. These UIRs are called **long** in the modern literature, cf., e.g., [12–18]. Analogously, we shall use for the cases when $d = d_{\max}$, i.e., (25a), the terminology of **semi−short** UIRs, introduced in [12,14], while the cases (25b, c, d) are also called **short** UIRs, cf., e.g., [13–18].

Next consider in more detail the UIRs at the four distinguished reduction points determining the list above:

$$\begin{aligned} d^1_{N1} &= 2 + 2j_2 + z + 2m_1 - 2m/N\,, && (26a)\\ d^2_{N1} &= z + 2m_1 - 2m/N\,, \quad (j_2 = 0)\,, && (26b)\\ d^3_{NN} &= 2 + 2j_1 - z + 2m/N\,, && (26c)\\ d^4_{NN} &= -z + 2m/N\,, \quad (j_1 = 0)\,. && (26d) \end{aligned}$$

The above reducibilities occur for the following odd roots, resp.:

$$\alpha_{3,4+N} = \gamma_2 + \gamma_4\,, \quad \alpha_{4,4+N} = \gamma_4\,, \quad \alpha_{15} = \gamma_1 + \gamma_3\,, \quad \alpha_{25} = \gamma_3\,. \qquad (27)$$

The corresponding singular vectors of $\tilde V^\Lambda$ are [8]:

$$\begin{aligned} \tilde v^1_{\rm odd} &= P_{3,4+N}|\widetilde{\Lambda}\rangle = \left(X^+_4 X^+_2 (h_2 - 1) - X^+_2 X^+_4 h_2\right)|\widetilde{\Lambda}\rangle = && (28a)\\ &= \left(2j_2 X^+_{3,4+N} - X^+_4 X^+_2\right)|\widetilde{\Lambda}\rangle\,, \qquad d = d^1_{N1}\,, \\ \tilde v^2_{\rm odd} &= X^+_4|\widetilde{\Lambda}\rangle\,, \qquad d = d^2_{N1}\,, && (28b)\\ \tilde v^3_{\rm odd} &= P_{15}|\widetilde{\Lambda}\rangle = \left(X^+_3 X^+_1 (h_1 - 1) - X^+_1 X^+_3 h_1\right)|\widetilde{\Lambda}\rangle = && (28c)\\ &= \left(2j_1 X^+_{15} - X^+_3 X^+_1\right)|\widetilde{\Lambda}\rangle\,, \qquad d = d^3_{NN}\,, \\ \tilde v^4_{\rm odd} &= X^+_3|\widetilde{\Lambda}\rangle\,, \qquad d = d^4_{NN}\,, && (28d) \end{aligned}$$

where $X^+_{3,4+N} = [X^+_2, X^+_4]$, $X^+_{15} = [X^+_1, X^+_3]$, h_1, $h_2 \in \mathcal{H}$ are Cartan generators corresponding to the roots γ_1, γ_2, (cf. [8]), and passing from the (28a), (28c), resp., to the next line we have used the fact that $h_2 v_0 = -2j_2 v_0$, $h_1 v_0 = -2j_1 v_0$, resp., consistently with (15b), (15a), resp.

For $j_1 = 0$, $j_2 = 0$, resp., the vector $\tilde v^1_s$, $\tilde v^2_s$, resp., is zero — cf. (24a), (24b), resp. However, then there is another independent singular vector of $\tilde V^\Lambda$ in both cases. For $j_1 = 0$ it corresponds to the sum of two roots: $\alpha_{15} + \alpha_{25}$ (which sum is not a root!) and is given by formula (D.1) of [8]:

$$\tilde v^{34} = X^+_3 X^+_1 X^+_3|\widetilde{\Lambda}\rangle = X^+_3 X^+_{15}|\widetilde{\Lambda}\rangle\,, \quad d = d^3_{NN}\,, \quad j_1 = 0\,. \qquad (29)$$

For $j_2 = 0$ there is a singular vector corresponding to the sum of two roots: $\alpha_{3,4+N} + \alpha_{4,4+N}$ (which sum is not a root) and is given in [8] (cf. the formula before (D.4)):

$$\tilde v^{12} = X^+_4 X^+_2 X^+_4|\widetilde{\Lambda}\rangle = X^+_4 X^+_{3,4+N}|\widetilde{\Lambda}\rangle\,, \quad d = d^1_{N1}\,, \quad j_2 = 0\,. \qquad (30)$$

From the expressions of the singular vectors follow, using (8), the explicit formulae for the corresponding invariant submodules I^β of the modules $\tilde{V}^\Lambda$ as follows:

$$
\begin{aligned}
I^1 &= U(\mathcal{G}^+) P_{3,4+N} \widetilde{|\Lambda\rangle} = U(\mathcal{G}^+) \left(X_4^+ X_2^+ (h_2 - 1) - X_2^+ X_4^+ h_2 \right) \widetilde{|\Lambda\rangle} = && \text{(31a)} \\
&= U(\mathcal{G}^+) \left(2j_2 X_{3,4+N}^+ - X_4^+ X_2^+ \right) \widetilde{|\Lambda\rangle} , \qquad d = d^1_{N1} , \quad j_2 > 0 , && \\
I^2 &= U(\mathcal{G}^+) X_4^+ \widetilde{|\Lambda\rangle} , \qquad d = d^2_{N1} , && \text{(31b)} \\
I^3 &= U(\mathcal{G}^+) P_{15} \widetilde{|\Lambda\rangle} = U(\mathcal{G}^+) \left(X_3^+ X_1^+ (h_1 - 1) - X_1^+ X_3^+ h_1 \right) \widetilde{|\Lambda\rangle} = && \text{(31c)} \\
&= U(\mathcal{G}^+) \left(2j_1 X_{15}^+ - X_3^+ X_1^+ \right) \widetilde{|\Lambda\rangle} , \qquad d = d^3_{NN} , \quad j_1 > 0 , && \\
I^4 &= U(\mathcal{G}^+) X_3^+ \widetilde{|\Lambda\rangle} , \qquad d = d^4_{NN} , && \text{(31d)} \\
I^{12} &= U(\mathcal{G}^+) \tilde{v}^{12} = X_4^+ X_2^+ X_4^+ \widetilde{|\Lambda\rangle} , \qquad d = d^1_{N1} , \quad j_2 = 0 , && \text{(31e)} \\
I^{34} &= U(\mathcal{G}^+) \tilde{v}^{34} = X_3^+ X_1^+ X_3^+ \widetilde{|\Lambda\rangle} , \qquad d = d^3_{NN} , \quad j_1 = 0 . && \text{(31f)}
\end{aligned}
$$

2.3 Structure of single−reducibility−condition Verma modules and UIRs

We discuss now the reducibility of Verma modules at the four distinguished points (26). We note a partial ordering of these four points:

$$
d^1_{N1} > d^2_{N1} , \quad d^3_{NN} > d^4_{NN} . \tag{32}
$$

Due to this ordering *at most two* of these four points may coincide.

In this Subsection we deal with the situations in which *no two* of the points in (26) coincide. There are four such situations involving UIRs:

$$
\begin{aligned}
d &= d_{\max} = d^1_{N1} = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN} , && \text{(33a)} \\
d &= d^2_{N1} > d^3_{NN} , \quad j_2 = 0 , && \text{(33b)} \\
d &= d_{\max} = d^3_{NN} = d^c \equiv 2 + 2j_1 - z + 2m/N > d^1_{N1} , && \text{(33c)} \\
d &= d^4_{NN} > d^1_{N1} , \quad j_1 = 0 , && \text{(33d)}
\end{aligned}
$$

where for future use we have introduced notation d^a, d^c.

We shall call these cases **single−reducibility−condition (SRC)** Verma modules or UIRs, depending on the context.

The factorized Verma modules $\tilde{V}^\Lambda$ with the unitary signatures from (33) have only one invariant (odd) submodule which has to be factorized in order to obtain the UIRs. These odd embeddings are given explicitly as:

$$
\tilde{V}^\Lambda \to \tilde{V}^{\Lambda+\beta} \tag{34}
$$

where we use the convention [7] that arrows point to the oddly embedded module, and there are the following cases for β:

$$\beta = \alpha_{3,4+N}\,, \quad \text{for (33a)}\,, \quad j_2 > 0\,, \tag{35a}$$
$$\beta = \alpha_{4,4+N}\,, \quad \text{for (33b)}\,, \tag{35b}$$
$$\beta = \alpha_{15}\,, \quad \text{for (33c)}\,, \quad j_1 > 0\,, \tag{35c}$$
$$\beta = \alpha_{25}\,, \quad \text{for (33d)}\,, \tag{35d}$$
$$\beta = \alpha_{3,4+N} + \alpha_{4,4+N}\,, \quad \text{for (33a)}\,, \quad j_2 = 0\,, \tag{35e}$$
$$\beta = \alpha_{15} + \alpha_{25}\,, \quad \text{for (33c)}\,, \quad j_1 = 0\,. \tag{35f}$$

This diagram gives the UIR L_Λ contained in $\tilde{V}^\Lambda$ as follows:

$$L_\Lambda = \tilde{V}^\Lambda / I^\beta\,, \tag{36}$$

where I^β is given by I^1, I^2, I^3, I^4, I^{12}, I^{34}, resp., (cf. (31)), in the cases (35a, b, c, d, e, f), resp.

It is useful to record the signatures of the shifted lowest weights, i.e., $\chi' = \chi(\Lambda+\beta)$. In fact, for future use we give the signature changes for arbitrary roots. The explicit formulae are [7,8]:

$$\begin{aligned} \beta &= \alpha_{3,N+5-k}\,, \qquad j_2 > 0\,, \quad r_{k-1} > 0\,, \\ \chi' &= \left[d+\tfrac{1}{2};\, j_1, j_2 - \tfrac{1}{2};\, z + \epsilon_N\,;\, r_1, \dots, r_{k-1} - 1, r_k + 1, \dots, r_{N-1}\right], \end{aligned} \tag{37a}$$

$$\begin{aligned} \beta &= \alpha_{4,N+5-k}\,, \qquad r_{k-1} > 0\,, \\ \chi' &= \left[d+\tfrac{1}{2};\, j_1, j_2 + \tfrac{1}{2};\, z + \epsilon_N\,;\, r_1, \dots, r_{k-1} - 1, r_k + 1, \dots, r_{N-1}\right], \end{aligned} \tag{37b}$$

$$\begin{aligned} \beta &= \alpha_{1,N+5-k}\,, \qquad j_1 > 0\,, \quad r_k > 0\,, \\ \chi' &= \left[d+\tfrac{1}{2};\, j_1 - \tfrac{1}{2}, j_2\,;\, z - \epsilon_N\,;\, r_1, \dots, r_{k-1} + 1, r_k - 1, \dots, r_{N-1}\right], \end{aligned} \tag{37c}$$

$$\begin{aligned} \beta &= \alpha_{2,N+5-k}\,, \qquad r_k > 0\,, \\ \chi' &= \left[d+\tfrac{1}{2};\, j_1 + \tfrac{1}{2}, j_2\,;\, z - \epsilon_N\,;\, r_1, \dots, r_{k-1} + 1, r_k - 1, \dots, r_{N-1}\right], \end{aligned} \tag{37d}$$

$$\begin{aligned} \beta_{12} &= \alpha_{3,4+N} + \alpha_{4,4+N}\,, \\ \chi'_{12} &= [d+1;\, j_1, 0;\, z + 2\epsilon_N\,;\, r_1 + 2, r_2, \dots, r_{N-1}]\,, \end{aligned} \tag{37e}$$

$$\begin{aligned} \beta_{34} &= \alpha_{15} + \alpha_{25}\,, \\ \chi'_{34} &= [d+1;\, 0, j_2\,;\, z - 2\epsilon_N\,;\, r_1, \dots, r_{N-2}, r_{N-1} + 2]\,. \end{aligned} \tag{37f}$$

$$\epsilon_N \equiv \frac{2}{N} - \frac{1}{2}\,. \tag{38}$$

For each fixed χ the lowest weight $\Lambda(\chi')$ fulfills the same odd reducibility condition as $\Lambda(\chi)$. The lowest weight $\Lambda(\chi'_{12})$ fulfils (33b), while the lowest weight $\Lambda(\chi'_{34})$ fulfils (33d).

2.4 Structure of double−reducibility−condition Verma modules and UIRs

We consider now the four situations in which *two* of the points in (26) coincide:

$$d = d_{\rm max} = d^{ac} \equiv 2 + j_1 + j_2 + m_1 = d^1_{N1} = d^3_{NN}\,, \tag{39a}$$
$$d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1\,, \quad j_1 = 0\,, \tag{39b}$$
$$d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1\,, \quad j_2 = 0\,, \tag{39c}$$
$$d = d^2_{N1} = d^4_{NN} = m_1\,, \quad j_1 = j_2 = 0\,. \tag{39d}$$

We shall call these **double−reducibility−condition (DRC)** Verma modules or UIRs. The odd embedding diagrams for the corresponding modules $\tilde V^\Lambda$ are:

$$\begin{array}{lll} \tilde V^{\Lambda+\beta'} & & \\ \uparrow & & \\ \tilde V^{\Lambda} & \longrightarrow & \tilde V^{\Lambda+\beta} \end{array} \tag{40}$$

$$(\beta,\beta') = (\alpha_{15}, \alpha_{3,4+N})\,, \quad \text{for (39a)}\,, \quad j_1 j_2 > 0\,, \tag{41a}$$
$$(\beta,\beta') = (\alpha_{15}, \alpha_{3,4+N} + \alpha_{3,4+N})\,, \quad \text{for (39a)}\,, \quad j_1 > 0\,, \quad j_2 = 0 \tag{41b}$$
$$(\beta,\beta') = (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N})\,, \quad \text{for (39a)}\,, \quad j_1 = 0\,, \quad j_2 > 0\,, \tag{41c}$$
$$(\beta,\beta') = (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N} + \alpha_{3,4+N})\,, \quad \text{for (39a)}\,, \quad j_1 = j_2 = 0\,, \tag{41d}$$
$$(\beta,\beta') = (\alpha_{25}, \alpha_{3,4+N})\,, \quad \text{for (39b)}\,, \quad j_2 > 0\,, \tag{41e}$$
$$(\beta,\beta') = (\alpha_{25}, \alpha_{3,4+N} + \alpha_{4,4+N})\,, \quad \text{for (39b)}\,, \quad j_2 = 0\,, \tag{41f}$$
$$(\beta,\beta') = (\alpha_{15}, \alpha_{4,4+N})\,, \quad \text{for (39c)}\,, \quad j_1 > 0\,, \tag{41g}$$
$$(\beta,\beta') = (\alpha_{15} + \alpha_{25}, \alpha_{4,4+N})\,, \quad \text{for (39c)}\,, \quad j_1 = 0\,, \tag{41h}$$
$$(\beta,\beta') = (\alpha_{25}, \alpha_{4,4+N})\,, \quad \text{for (39d)}\,. \tag{41i}$$

This diagram gives the UIR L_Λ contained in $\tilde V^\Lambda$ as follows:

$$L_\Lambda = \tilde V^\Lambda / I^{\beta,\beta'}\,, \quad I^{\beta,\beta'} = I^\beta \cup I^{\beta'}\,, \tag{42}$$

where I^β, $I^{\beta'}$ are given in (31), accordingly to the cases in (41).

Naturally, the two odd embeddings in (40) are the combination of the different cases of (34). However, (40) is a piece of a richer picture (given in [7]) which is important for the character formulae. For the lack of space we omit it referring to [1].

3 Character formulae of positive energy UIRs

3.1 Character formulae: generalities

In the beginning of this subsection we follow [19]. Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat\Delta$, simple root system $\hat\pi$. Let Γ, (resp. Γ_+), be the set of all integral, (resp. integral dominant), elements of

$\hat{\mathcal{H}}^*$, i.e., $\lambda \in \hat{\mathcal{H}}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i, ($\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i,\alpha_i)$). Let V be a lowest weight module with lowest weight Λ and lowest weight vector v_0. It has the following decomposition:

$$V = \underset{\mu\in\Gamma_+}{\oplus} V_\mu \,, \quad V_\mu = \{u \in V | Hu = (\lambda+\mu)(H)u \ \forall H \in \mathcal{H}\} \,. \tag{43}$$

(Note that $V_0 = \mathbb{C} v_0$.) Let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu\in\mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathcal{H}^* | \mu \geq \lambda\}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1$, $e(\mu)e(\nu) = e(\mu+\nu)$.

Then the (formal) character of V is defined by:

$$\mathrm{ch}\, V = \sum_{\mu\in\Gamma_+} (\dim V_\mu) e(\Lambda+\mu) = e(\Lambda) \sum_{\mu\in\Gamma_+} (\dim V_\mu) e(\mu) \,. \tag{44}$$

The character formula for Verma modules is [19]:

$$\mathrm{ch}\, V^\Lambda = e(\Lambda) \prod_{\alpha\in\Delta^+} (1 - e(\alpha))^{-1} \,. \tag{45}$$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha \,, \quad \lambda \in \hat{\mathcal{H}}^* \,, \quad \alpha \in \hat{\Delta} \,. \tag{46}$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $\alpha_i \in \hat{\pi}$. The Weyl character formula for the finite–dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form:

$$\mathrm{ch}\, L_\Lambda = \sum_{w\in W} (-1)^{\ell(w)} \,\mathrm{ch}\, V^{w\cdot\Lambda} \,, \quad \Lambda \in -\Gamma_+ \,, \tag{47}$$

where the dot $\cdot$ action is defined by $w \cdot \lambda = w(\lambda - \rho) + \rho$.

In the case of basic classical Lie superalgebras (except $osp(1/2N)$) the character formula for Verma modules is [10]:

$$\mathrm{ch}\, V^\Lambda = e(\Lambda) \left(\prod_{\alpha\in\Delta_{\bar 0}^+} (1 - e(\alpha))^{-1} \right) \left(\prod_{\alpha\in\Delta_{\bar 1}^+} (1 + e(\alpha)) \right) . \tag{48}$$

Note that this may be written as:

$$\mathrm{ch}\, V^\Lambda = \mathrm{ch}\, V_0^\Lambda \,\mathrm{ch}\, \hat{V}^\Lambda \,, \quad ch\ \hat{V}^\Lambda \equiv \prod_{\alpha\in\Delta_{\bar 1}^+} (1 + e(\alpha)) \,, \tag{49}$$

where $\mathrm{ch}\, V_0^\Lambda$ is the character of the restriction of V^Λ to the even subalgebra, and $\hat{V}^\Lambda \equiv \left(U(\mathcal{G}_+^{\mathbb{C}})/(\mathcal{G}_+^{\mathbb{C}})_{(0)}\right) |\widetilde{\Lambda}\rangle$. Obviously, $\hat{V}^\Lambda$ may be viewed as the result of all possible application of the $4N$ odd generators $X^+_{a,4+k}$ on $|\Lambda\rangle$. Thus, $\hat{V}^\Lambda$ has 2^{4N} states (including the vacuum). Explicitly, the basis of $\hat{V}^\Lambda$ is [9]:

$$\Psi_{\bar\varepsilon} = \left(\prod_{k=N}^{1} (X^+_{1,4+k})^{\varepsilon_{1,4+k}}\right)\left(\prod_{k=N}^{1} (X^+_{2,4+k})^{\varepsilon_{2,4+k}}\right) \times \\ \times \left(\prod_{k=1}^{N} (X^+_{3,4+k})^{\varepsilon_{3,4+k}}\right)\left(\prod_{k=1}^{N} (X^+_{4,4+k})^{\varepsilon_{4,4+k}}\right) |\widetilde{\Lambda}\rangle\,, \quad \varepsilon_{aj} = 0,1\,, \tag{50}$$

where $\bar\varepsilon$ denotes the set of all ε_{ij}.

The odd null conditions entwine with the even null conditions as we shall see. The even null conditions carry over from the even null conditions (24) of $\tilde{V}^\Lambda$:

$$(X_1^+)^{1+2j_1}|\Lambda\rangle = 0\,, \tag{51a}$$
$$(X_2^+)^{1+2j_2}|\Lambda\rangle = 0\,, \tag{51b}$$
$$(X_j^+)^{1+r_{N+4-j}}|\Lambda\rangle = 0\,, \quad j = 5,\ldots,N+3\,, \tag{51c}$$

where by $|\Lambda\rangle$ we denote the lowest weight vector of the UIR L_Λ.

For future use we introduce notation for the levels of the different chiralities ε_i and the overall level ε

$$\varepsilon_i = \sum_{k=1}^{N} \varepsilon_{i,4+k}\,, \quad i = 1,2,3,4\,, \quad \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4\,. \tag{52}$$

3.2 Character formulae for the long UIRs

As we mentioned if $d > d_{\max}$ there are no further reducibilities, and the UIRs $L_\Lambda = \tilde{V}^\Lambda$ are called *long* since $\hat{L}_\Lambda$ may have the maximally possible number of states 2^{4N} (including the vacuum state). However, the actual number of states may be less than 2^{4N} states due to the fact that — depending on the values of j_a and r_k — not all actions of the odd generators on the vacuum would be allowed. The latter is obvious from formulae (37). Using the latter we can give the resulting signature of the state $\Psi_{\bar\varepsilon}$:

$$\chi(\Psi_{\bar\varepsilon}) = \big[d + \tfrac{1}{2}\varepsilon\,;\, j_1 + \tfrac{1}{2}(\varepsilon_2 - \varepsilon_1),\, j_2 + \tfrac{1}{2}(\varepsilon_4 - \varepsilon_3)\,;\, z + \epsilon_N(\varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_2)\,; \\ \ldots, r_i + \varepsilon_{1,N+4-i} - \varepsilon_{1,N+5-i} + \varepsilon_{2,N+4-i} - \varepsilon_{2,N+5-i} - \\ - \varepsilon_{3,N+4-i} + \varepsilon_{3,N+5-i} - \varepsilon_{4,N+4-i} + \varepsilon_{4,N+5-i}\,, \ldots\big]\,. \tag{53}$$

Thus, only if $j_1, j_2 \geq N/2$ and $r_i \geq 4$ (for all i) the number of states is 2^{4N} [3], and the character formula is:

$$\mathrm{ch}\, L_\Lambda = \mathrm{ch}\, L^0_\Lambda\, \mathrm{ch}\, \hat{V}^\Lambda\,, \tag{54a}$$
$$j_1, j_2 \geq N/2\,, \quad r_i \geq 4\,, \quad i = 1,\ldots,N-1\,, \tag{54b}$$

where $\mathrm{ch}\, L^0_\Lambda$ denotes the character of the restriction of L_Λ to the even subalgebra.

The general formula for $\mathrm{ch}\, L_\Lambda$ shall be written in a similar fashion:

$$\mathrm{ch}\, L_\Lambda = \mathrm{ch}\, L^0_\Lambda\, \mathrm{ch}\, \hat{L}_\Lambda\,. \tag{55}$$

Moreover, from now on we shall write only the formulae for $\mathrm{ch}\, \hat{L}_\Lambda$. Thus, formula (54) may be written equivalently as:

$$\mathrm{ch}\, \hat{L}_\Lambda = \mathrm{ch}\, \hat{V}^\Lambda\,, \quad j_1, j_2 \geq N/2\,, \quad r_i \geq 4\ \forall\, i\,. \tag{56}$$

If the auxiliary conditions (54b) are not fulfilled then a careful analysis is necessary. To simplify the exposition we classify the states by the following quantities:

$$\begin{aligned}
\varepsilon^c_j &\equiv \varepsilon_1 - \varepsilon_2\,,\\
\varepsilon^a_j &\equiv \varepsilon_3 - \varepsilon_4\,,\\
\varepsilon^i_r &\equiv \varepsilon_{1,5+i} + \varepsilon_{2,5+i} + \varepsilon_{3,4+i} + \varepsilon_{4,4+i} - \varepsilon_{1,4+i} - \varepsilon_{2,4+i} - \varepsilon_{3,5+i} - \varepsilon_{4,5+i}\,,\\
&\quad i = 1, \ldots, N-1\,.
\end{aligned} \tag{57}$$

This gives the following necessary conditions on ε_{ij} for a state to be allowed:

$$\varepsilon^c_j \leq 2j_1\,, \tag{58a}$$

$$\varepsilon^a_j \leq 2j_2\,, \tag{58b}$$

$$\varepsilon^i_r \leq r_{N-i}\,, \quad i = 1, \ldots, N-1\,. \tag{58c}$$

These conditions are also sufficient only for $N = 1$. The exact conditions are:
Criterion: The necessary and sufficient conditions for the state $\Psi_{\bar\varepsilon}$ of level ε to be allowed are that conditions (58) are fulfilled and that the state is a descendant of an allowed state of level $\varepsilon - 1$. $\diamondsuit$

The second part of the Criterion will take care first of all of chiral (or anti-chiral) states when some ε_{aj} contribute to opposing sides of the inequalities in (58a) and (58c), (or (58b) and (58c)). This happens for $j_1 = r_i = 0$, (or $j_2 = r_i = 0$).

We shall give now the most important such occurrences. Take first chiral states, i.e., all $\varepsilon_{3,4+k} = \varepsilon_{3,4+k} = 0$. Fix $i = 1, \ldots, N-1$. It is easy to see that the following states are not allowed [1]:

$$\begin{aligned}
\psi_{ij} &= \phi_{ij}|\Lambda\rangle = X^+_{1,i+4} X^+_{2,i+5} X^+_{a_1,i+6} \cdots X^+_{a_{j-1},i+4+j}|\Lambda\rangle\,, \quad a_n = 1, 2\,,\\
&\quad j = 1, \ldots, N-i\,, \quad j_1 = r_{N-i} = \cdots = r_{N-i-j+1} = 0\,,\\
&\quad \text{in addition, for } N > 2\,,\ i > 1 \text{ holds } r_{N-i+1} \neq 0\,.
\end{aligned} \tag{59}$$

Consider now anti-chiral states, i.e., such that $\varepsilon_{1,4+k} = \varepsilon_{2,4+k} = 0$, for all $k = 1, \ldots, N$. Fix $i = 1, \ldots, N-1$. Then the following states are not allowed:

$$\begin{aligned}
\psi'_{ij} &= \phi'_{ij}|\Lambda\rangle = X^+_{3,i+5} X^+_{4,i+4} X^+_{b_1,i+3} \cdots X^+_{b_{j-1},i+5-j}|\Lambda\rangle\,, \quad b_n = 3, 4\,,\\
&\quad j = 1, \ldots, i\,, \quad j_2 = r_{N-i} = \cdots = r_{N-i+j-1} = 0\,,\\
&\quad \text{in addition, for } N > 2\,,\ i > 1 \text{ holds } r_{N-i-1} \neq 0\,.
\end{aligned} \tag{60}$$

Furthermore, any combinations of ϕ_{ij} and $\phi'_{i'j'}$ are not allowed.

Note that for $N \geq 4$ the states in (59), (60) do not exhaust the states forbidden by our Criterion. For example, for $N=4$ there are the following forbidden states:

$$\psi_4 = \phi_4|\Lambda\rangle = X^+_{28}X^+_{17}X^+_{16}X^+_{25}|\Lambda\rangle\,, \quad j_1 = r_1 = r_2 = r_3 = 0\,, \tag{61a}$$
$$\psi'_4 = \phi'_4|\Lambda\rangle = X^+_{45}X^+_{36}X^+_{37}X^+_{48}|\Lambda\rangle\,, \quad j_2 = r_1 = r_2 = r_3 = 0\,. \tag{61b}$$

Summarizing the discussion so far, the general character formula may be written as follows:

$$\begin{aligned}
\mathrm{ch}\,\hat{L}_\Lambda &= \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}\,, \qquad d > d_{\max}\,, \\
\mathcal{R} &= e(\hat{V}^\Lambda_{\rm excl}) = \sum_{\substack{\text{excluded}\\ \text{states}}} e(\Psi_{\bar{\varepsilon}})\,, \\
e(\Psi_{\bar{\varepsilon}}) &= \left(\prod_{k=N}^{1} e(\alpha_{1,4+k})^{\varepsilon_{1,4+k}}\right)\left(\prod_{k=N}^{1} e(\alpha_{2,4+k})^{\varepsilon_{2,4+k}}\right)\times \\
&\quad\times\left(\prod_{k=1}^{N} e(\alpha_{3,4+k})^{\varepsilon_{3,4+k}}\right)\left(\prod_{k=1}^{N} e(\alpha_{4,4+k})^{\varepsilon_{4,4+k}}\right),
\end{aligned} \tag{62}$$

where the counter–terms denoted by $\mathcal{R}$ are determined by $\hat{V}^\Lambda_{\rm excl}$ which is the collection of all states (i.e., collection of ε_{jk}) which violate the conditions (58), or are impossible in the sense of (59) and/or (60). Of course, each excluded state is accounted for only once even if it is not allowed for several reasons.

Finally, we consider two important conjugate special cases. First, the anti-chiral sector of R–symmetry scalars with $j_2 = 0$. Taking into account (58b, c) and our Criterion it is easy to see that the appearance of the generators $X^+_{3,4+k}$ is restricted as follows. The generator $X^+_{3,4+N}$ may appear only in the state

$$X^+_{3,4+N}X^+_{4,4+N}|\Lambda\rangle \tag{63}$$

and its descendants. The generator $X^+_{3,3+N}$ may only appear either in states descendant to the state (63) or in the state

$$X^+_{3,3+N}X^+_{4,4+N}|\Lambda\rangle \tag{64}$$

and its descendants including only generators $X^+_{a,4+N-\ell}$, $a = 3,4$, $\ell > 1$. Further, fix ℓ such that $1 < \ell \leq N-1$. The generator $X^+_{3,4+N-\ell}$ may only appear either in states containing generators $X^+_{3,4+N-j}$, where $0 \leq j < \ell$, or in the state

$$X^+_{3,4+N-\ell}X^+_{4,5+N-\ell}X^+_{4,6+N-\ell}\cdots X^+_{4,4+N}|\Lambda\rangle \tag{65}$$

and its descendants including only generators $X^+_{a,4+N-\ell'}$, $a = 3,4$, $\ell' > \ell$.

The anti-chiral part of the basis is further restricted. Namely, there are only N anti-chiral states that can be built from the generators $X^+_{4,4+k}$ alone:

$$X^+_{4,5+N-k}X^+_{4,6+N-k}\cdots X^+_{4,4+N}|\Lambda\rangle\,, \quad k = 1,\ldots,N\,, \quad j_2 = r_i = 0\ \forall\, i\,. \tag{66}$$

This follows from (58c) which for such states becomes $\varepsilon_{4,4+N-i} \le \varepsilon_{4,5+N-i}$ for $i = 1, \dots, N-1$.

The chiral sector of R–symmetry scalars with $j_1 = 0$ is obtained from the above by conjugation.

3.3 Character formulae of SRC UIRs

•a $\quad d = d^1_{N1} = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN}$.

• Let first $j_2 > 0$. In these semi–short SRC cases holds the odd null condition following from (28a):

$$P_{3,4+N}|\Lambda\rangle = \left(2j_2 X^+_{3,4+N} - X^+_4 X^+_2\right)|\Lambda\rangle = 0\,. \tag{67}$$

Clearly, condition (67) means that the generator $X^+_{3,4+N}$ is eliminated from the basis that is built on the lowest weight vector $|\Lambda\rangle$. Thus, for $N = 1$ and if $r_1 > 0$ for $N > 1$ the character formula is:

$$\mathrm{ch}\,\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta^+_{\bar 1}\\ \alpha\neq\alpha_{3,4+N}}} (1 + e(\alpha)) - \mathcal{R}\,, \quad j_2 r_1 > 0\,. \tag{68}$$

There are no counter–terms when $j_1 \ge N/2$, $j_2 \ge (N-1)/2$ and $r_i \ge 4$ (for all i), and then the number of states is 2^{4N-1}.

When there are no counter–terms (also for the complex $sl(4/N)$ case) this formula follows easily from (36). Indeed, in the case at hand $I^\beta = I^1$, (cf. (31a)); then from $L_\Lambda = \tilde{V}^\Lambda/I^1$ follows $\hat{L}_\Lambda = \hat{V}^\Lambda/\hat{I}^1$ and:

$$\mathrm{ch}\,\hat{L}_\Lambda = \mathrm{ch}\,\hat{V}^\Lambda - \mathrm{ch}\,\hat{I}^1\,, \tag{69}$$

where $\hat{I}^1$ is the projection of I^1 to the odd sector. Naively, the character of $\hat{I}^1$ should be given by the character of $\hat{V}^{\Lambda+\alpha_{3,4+N}}$, however, as discussed in general — cf. (8), I^1 is smaller than $\hat{V}^{\Lambda+\alpha_{3,4+N}}$ and its character is given with a prefactor:

$$\mathrm{ch}\,\hat{I}^1 = \frac{1}{1 + e(\alpha_{3,4+N})}\,\mathrm{ch}\,\hat{V}^{\Lambda+\alpha_{3,4+N}} = \frac{e(\alpha_{3,4+N})}{1 + e(\alpha_{3,4+N})}\,\mathrm{ch}\,\hat{V}^\Lambda\,. \tag{70}$$

Now (68) (with $\mathcal{R} = 0$) follows from the combination of (69) and (70).

Formula (68) may also be described by using the odd reflection (10) with $\beta = \alpha_{3,4+N}$:

$$\mathrm{ch}\,\hat{L}_\Lambda = \mathrm{ch}\,\hat{V}^\Lambda - \frac{1}{1 + e(\alpha_{3,4+N})}\,\mathrm{ch}\,\hat{V}^{\hat{s}_{\alpha_{3,4+N}}\cdot\Lambda} - \mathcal{R} = \tag{71a}$$

$$= \sum_{\hat{s}\in\hat{W}_{\alpha_{3,4+N}}} (-1)^{\ell(\hat{s})}\,\hat{s}\cdot\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}\,, \tag{71b}$$

where $\hat{W}_\beta \equiv \{1, \hat{s}_\beta\}$ is a two–element semi-group restriction of $\tilde{W}_\beta$, and we have formalized further by introducing notation for the action of an odd reflection on characters:

$$\hat{s}_\beta \cdot \mathrm{ch}\, V^\Lambda = \frac{1}{1+e(\beta)}\ \mathrm{ch}\, V^{\hat{s}_\beta \cdot \Lambda} = \frac{1}{1+e(\beta)}\ \mathrm{ch}\, V^{\Lambda+\beta} = \frac{e(\beta)}{1+e(\beta)}\ \mathrm{ch}\, V^\Lambda\,. \tag{72}$$

It is natural to introduce the restriction $\hat{W}_\beta$ since only the identity element of $\tilde{W}_\beta$ and the generator $\hat{s}_\beta$ act nontrivially because the action $\hat{s}_\beta$ on characters is nilpotent:

$$(\hat{s}_\beta)^2 \cdot \mathrm{ch}\, V^\Lambda = 0\,. \tag{73}$$

In fact, we shall need more general formula for the action of odd reflections on polynomials $\mathcal{P}$ from $E(\mathcal{H}^*)$. Thus, instead of (72) we shall define the action of $\hat{s}_\beta$ on $\mathcal{P}$ as a homogeneity operator treating $e(\beta)$ as a variable:

$$\hat{s}_\beta \cdot \mathcal{P} \equiv e(\beta)\, \frac{\partial}{\partial e(\beta)}\, \mathcal{P}\,, \tag{74}$$

where β may be a root or the sum of roots. Obviously, if $\mathcal{P}$ is a monomial which contains a multiplicative factor $1+e(\beta)$ the action (74) is equivalent to (72).

In particular, we shall show that in many cases character formulae (68), (71) may be written as follows:

$$\mathrm{ch}\, \hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_\beta} (-1)^{\ell(\hat{s})} \hat{s} \cdot \Big(\mathrm{ch}\, \hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\Big)\,, \tag{75}$$

where $\mathcal{R}_{\mathrm{long}}$ represents the counter–terms for the long superfields for the same values of j_1 and r_i as Λ, while the value of j_2 is zero when j_2 from Λ is zero, otherwise it has to be the generic value $j_2 \geq N/2$.

Writing (68) as (71) (or (75)) may look as a complicated way to describe the cancellation of a factor from the character formula for $\hat{V}^\Lambda$, however, first of all it is related to the structure of $\tilde{V}^\Lambda$ given by (36), and furthermore may be interpreted — when there are no counter–terms — as the following decomposition:

$$\hat{V}^\Lambda = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta}\,, \tag{76}$$

for $\beta = \alpha_{3,4+N}$. Indeed, for generic signatures $\hat{L}_{\Lambda+\beta}$ is isomorphic to $\hat{L}_\Lambda$ as a vector space (this is due to the fact that $V^{\Lambda+\beta}$ has the same reducibilities as V^Λ, cf. Section 2), they differ only by the vacuum state. Thus, when there are no counter–terms, both $\hat{L}_\Lambda$ and $\hat{L}_{\Lambda+\beta}$ have the same 2^{4N-1} states.

It is more important that there is a similar decomposition valid for many cases beyond the generic, i.e., we have:

$$\Big(\hat{L}_{\mathrm{long}}\Big)_{|_{d=d^a}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}}\,, \quad N=1 \ \text{ or } \ r_1>0 \text{ for } N>1\,, \tag{77}$$

where $\hat{L}_{\mathrm{long}}$ is a long superfield with the same values of j_1 and r_i as Λ, while the value of j_2 has to be specified, and equality is as vector spaces.

For $N > 1$ there are possible additional truncations of the basis. To make the exposition easier we need additional notation. Let i_0 be an integer such that $0 \le i_0 \le N-1$, and $r_i = 0$ for $i \le i_0$, and if $i_0 < N-1$ then $r_{i_0+1} > 0$.

Let now $N > 1$ and $i_0 > 0$; then the generators $X^+_{3,4+N-i}$, $i = 1, \ldots, i_0$, are eliminated from the basis. This follows from the following recursive null conditions:

$$P_{3,4+N-i}|\Lambda\rangle = \left(2j_2 X^+_{3,4+N-i} - X^+_{4,4+N-i}X^+_2\right)|\Lambda\rangle = 0\,, \quad i \le i_0\,. \tag{78}$$

From the above follows that for $i_0 > 0$ the decomposition (77) can not hold. Indeed, the generators $X^+_{3,4+N-i}$, $i = 1, \ldots, i_0$, are eliminated from the irrep $\hat{L}_\Lambda$ due to the fact that we are at a reducibility point, but there is no reason for them to be eliminated from the long superfield. Certainly, some of these generators are present in the second term $\hat{L}_{\Lambda+\alpha_{3,4+N}}$ in (77), but that would be only those which in the long superfield were in states of the kind: $\Phi\, X^+_{3,4+N}|\Lambda\rangle$, and, certainly, such states do not exhaust the occurrence of the discussed generators in the long superfield. Symbolically, instead of the decomposition (77) we shall write:

$$\left(\hat{L}_{\rm long}\right)_{|_{d=d^a}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}} \oplus \hat{L}'_\Lambda\,, \quad N > 1\,, \quad i_0 > 0\,, \tag{79}$$

where we have represented the excess states by the last term with prime. With the prime we stress that this is not a genuine irrep, but just a book–keeping device. Formulae as (79) in which not all terms are genuine irreps shall be called *quasi-decompositions*.

The corresponding character formula is:

$$\begin{aligned}
{\rm ch}\,\hat{L}_\Lambda &= \prod_{\substack{\alpha\in\Delta^+_{\bar 1} \\ \alpha\neq\alpha_{3,5+N-k} \\ k=1,\ldots,1+i_0}} (1+e(\alpha)) - \mathcal{R} = && \text{(80a)}\\
&= \sum_{\hat{s}\in\hat{W}^a_{i_0}} (-1)^{\ell(\hat{s})}\hat{s}\cdot {\rm ch}\,\hat{V}^\Lambda - \mathcal{R} = && \text{(80b)}\\
&= \sum_{\hat{s}\in\hat{W}^a_{i_0}} (-1)^{\ell(\hat{s})}\hat{s}\cdot \left({\rm ch}\,\hat{V}^\Lambda - \mathcal{R}_{\rm long}\right), && \text{(80c)}\\
\hat{W}^a_{i_0} &\equiv \hat{W}_{\alpha_{3,N+4}} \times \hat{W}_{\alpha_{3,N+3}} \times \cdots \times \hat{W}_{\alpha_{3,N+4-i_0}}\,. && \text{(80d)}
\end{aligned}$$

The restrictions (58) used to determine the counter–terms are, of course, with $\varepsilon_{3,5+N-k} = 0$, $k = 1, \ldots, 1+i_0$. Formulae (68), (71), (75) are special cases of (80a, b, c), resp., for $i_0 = 0$. The maximal number of states in $\hat{L}_\Lambda$ is 2^{4N-1-i_0}. This is the number of states that is obtained from the action of the Weyl group $\hat{W}^a_{i_0}$ on ${\rm ch}\,\hat{V}^\Lambda$, while the actual counter–term is obtained from the action of the Weyl group on $\mathcal{R}_{\rm long}$.

In the extreme case of R–symmetry scalars: $i_0 = N-1$, i.e., $r_i = 0$, $i = 1, \ldots, N-1$, or, equivalently, $m_1 = 0 = m$, all the N generators $X^+_{3,4+k}$ are eliminated. The character formula is again (80) taken with $i_0 = N-1$.

• Let now $j_2 = 0$. Then all null conditions above follow from (24b), so these conditions do not mean elimination of the mentioned vectors. As we know in this situation we have the singular vector (30) which leads to the following null condition:

$$X^+_{3,4+N} X^+_{4,4+N} |\Lambda\rangle = X^+_4 X^+_2 X^+_4 |\Lambda\rangle = 0\,. \tag{81}$$

The state in (81) and all of its 2^{4N-2} descendants are zero for any N. Thus, the character formula is similar to (71), but with $\alpha_{3,4+N}$ replaced by $\beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}$:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_{\beta_{12}}} (-1)^{\ell(\hat{s})}\hat{s}\cdot\left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\right), \quad N = 1 \text{ or } r_1 > 0\,, \tag{82}$$

where $\hat{W}_{\beta_{12}} \equiv \{1, \beta_{12}\}$. Note that for $N = 1$ formula (82) is equivalent to (68).

Here holds a decomposition similar to (77):

$$\left(\hat{L}_{\mathrm{long}}\right)_{|d=d^a} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{12}}\,, \quad N = 1 \text{ or } r_1 > 0 \text{ for } N > 1\,, \tag{83}$$

where $\hat{L}_{\mathrm{long}}$ is with the same values of j_1, $j_2(=0)$, r_i as Λ. Note, however, that the UIR $\hat{L}_{\Lambda+\beta_{12}}$ belongs to type **b** below.

There are more eliminations for $N > 1$ when $i_0 > 0$. For instance we can show that all states obtained as in (65) considered for $\ell = 1, \ldots, i_0$ are not allowed [1]. From this follows that if $i_0 > 0$ the decomposition (83) does not hold. Instead, there is a quasi-decomposition similar to (79).

We can be more explicit in the case when all $r_i = 0$. In that case all the vectors $X^+_{3,5+N-k}$ are eliminated from all anti-chiral states [1].

The anti-chiral part of the basis is further restricted. As we know, when $j_2 = r_i = 0\ \forall\, i$, there are only N anti-chiral states that can be built from the generators $X^+_{4,4+k}$ alone, given in (66). Thus the corresponding character formula is:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{k=1}^{N}\prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \prod_{\substack{\alpha\in\Delta^+_{\bar{1}}\\ \varepsilon_1+\varepsilon_2>0}} (1 + e(\alpha)) - \mathcal{R}\,, \quad j_2 = r_i = 0\ \forall\, i\,. \tag{84}$$

•**b** $\quad d = d^2_{N1} = z + 2m_1 - 2m/N > d^3_{NN}$, $j_2 = 0$.
In these short single–reducibility–condition cases holds the odd null condition (following from the singular vector (28b))

$$X^+_4 |\Lambda\rangle = X^+_{4,4+N} |\Lambda\rangle = 0\,. \tag{85}$$

Since $j_2 = 0$ from (24b) and (85) follows the additional null condition:

$$X^+_{3,4+N} |\Lambda\rangle = [X^+_2, X^+_4]\,|\Lambda\rangle = 0\,. \tag{86}$$

For $N > 1$ and $r_1 > 2$ each of these UIRs enters as the second term in decomposition (83), when the first term is an UIR of type **a** with $j_2 = 0$, as explained above.

Further, for $N > 1$ there are additional recursive null conditions if $r_i = 0$, $i \le i_0$ which follow from (24c) and (86):

$$X^+_{3,4+N-i}|\Lambda\rangle = [X^+_{3,5+N-i}, X^+_{4+N-i}]\,|\Lambda\rangle = 0\,, \quad r_j = 0\,, \quad 1 \le j \le i \le i_0\,, \tag{87a}$$
$$X^+_{4,4+N-i}|\Lambda\rangle = [X^+_{4,5+N-i}, X^+_{4+N-i}]\,|\Lambda\rangle = 0\,, \quad r_j = 0\,, \quad 1 \le j \le i \le i_0\,. \tag{87b}$$

Thus, $2(1+i_0)$ generators $X^+_{3,5+N-k}$, $X^+_{4,5+N-k}$, $k = 1, \dots, 1+i_0$, are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-2-2i_0}$.

The corresponding character formula is:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^b_{i_0}} (-1)^{\ell(\hat{s})}\hat{s}\cdot \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}\,, \tag{88a}$$

$$\begin{aligned}\hat{W}^b_{i_0} \equiv\ & \hat{W}_{\alpha_{3,N+4}} \times \hat{W}_{\alpha_{3,N+3}} \times \cdots \times \hat{W}_{\alpha_{3,N+4-i_0}} \times \\ & \times \hat{W}_{\alpha_{4,N+4}} \times \hat{W}_{\alpha_{4,N+3}} \times \cdots \times \hat{W}_{\alpha_{4,N+4-i_0}}\,, \qquad j_2 = r_i = 0\,, \quad i \le i_0\,,\end{aligned} \tag{88b}$$

where determining the counter–terms we use $\varepsilon_{a,4+k} = 0$, $a = 3,4$, $k = 1, \dots, 1+i_0$.

In the case of R–symmetry scalars ($i_0 = N-1$) we have:

$$X^+_{3,4+k}|\Lambda\rangle = 0\,, \quad X^+_{4,4+k}|\Lambda\rangle = 0\,, \qquad k = 1, \dots, N\,, \quad r_i = 0\;\forall\, i\,. \tag{89}$$

The character formula is (88) taken with $1 + i_0 = N$. These UIRs should be called chiral since all anti-chiral generators are eliminated.

•**c** $\quad d = d^3_{NN} = d^c \equiv 2 + 2j_1 - z + 2m/N > d^1_{N1}$.
•**d** $\quad d = d^4_{NN} = -z + 2m/N > d^1_{N1}$, $j_1 = 0$.
These cases are conjugate to the cases **a**, **b**, resp. All results may be obtained by the substitutions: $j_1 \longleftrightarrow j_2$, $r_i \longleftrightarrow r_{N-i}$, $z \longleftrightarrow -z$, $\alpha_{a,4+k} \longleftrightarrow \alpha_{4-a,N+5-k}$, $a = 1,2$, $k = 1, \dots, N$, and so we shall omit them here, cf. [1].

3.4 Character formulae of DRC UIRs

Let first $N > 1$ and $r_1 r_N - 1 > 0$, (i.e., $i_0 = i'_0 = 0$). Then holds the following character formula:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_{\beta,\beta'}} (-1)^{\ell(\hat{s})}\hat{s}\cdot \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R} = \tag{90a}$$

$$\begin{aligned} &= \mathrm{ch}\,\hat{V}^\Lambda - \frac{1}{1+e(\beta)}\ \mathrm{ch}\,\hat{V}^{\Lambda+\beta} - \frac{1}{1+e(\beta')}\ \mathrm{ch}\,\hat{V}^{\Lambda+\beta'} + \\ &\quad + \frac{1}{(1+e(\beta))(1+e(\beta'))}\ \mathrm{ch}\,\hat{V}^{\Lambda+\beta+\beta'} - \mathcal{R}\,, \end{aligned} \tag{90b}$$

$$\hat{W}_{\beta,\beta'} \equiv \hat{W}_\beta \times \hat{W}_{\beta'}\,. \tag{90c}$$

The above formula is proved similarly to what we had in the SRC cases, however, it takes into account the richer structure given explicitly already in the paper [7].

The proof is as follows [1]: The two terms with minus sign on the first line of (90b) take into account the factorization of the oddly embedded submodules I^β, $I^{\beta'}$, cf. (42). The nontrivial moment is the contribution of the module $\hat{V}^{\Lambda+\beta+\beta'}$. It is oddly embedded in $\hat{V}^\Lambda$ via both submodules $\hat{V}^{\Lambda+\beta}$ and $\hat{V}^{\Lambda+\beta'}$, [7], and its contribution is taken out *two times* — once with I^β, and a second time with $I^{\beta'}$. Thus, we need the term with plus sign on the second line of (90b) to restore its contribution once. We can not apply the same kind of arguments for $N = 1$, nevertheless, formula (90) holds also then for the case (41a), cf. Appendix A.1. of [1].

•ac $\quad d = d_{\max} = d^1_{N1} = d^3_{NN} = d^{ac} \equiv 2 + j_1 + j_2 + m_1 .$

In these semi–short DRC cases hold the null condition (67) and its conjugate. In addition, for $N > 1$ if $r_i = 0$, $i = 1, \ldots, i_0$, holds (78) and if $r_{N-i} = 0$, $i = 1, \ldots, i'_0$, holds the conjugate to (78).

There are two basic situations. The first is when $i_0 + i'_0 \leq N - 2$. This means that not all r_i are zero and all eliminations are as described separately for cases **•a** and **•c**. These semi–short UIRs may be called Grassmann–analytic following [14], since odd generators from different chiralities are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-2-i_0-i'_0}$.

The second is when $i_0 + i'_0 \leq N - 2$ does not hold which means that all r_i are zero, and in fact we have $i_0 = i'_0 = N - 1$ and all generators $X^+_{1,4+k}$ and $X^+_{3,4+k}$ are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is 2^{2N}.

• For $j_1 j_2 > 0$ the corresponding character formulae are combinations of (80) and its conjugate [1]:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in \hat{W}^{ac}_{i_0,i'_0}} (-1)^{\ell(\hat{s})}\,\hat{s}\cdot\Big(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\Big)\,, \tag{91a}$$

$$\begin{aligned} \hat{W}^{ac}_{i_0,i'_0} &\equiv \hat{W}^a_{i_0} \times \hat{W}^c_{i'_0}\,, \qquad j_1 j_2 > 0\,, \\ &\text{either}\ \ i_0 + i'_0 \leq N - 2\,, \\ &\qquad\qquad r_i = 0\,,\quad i = 1, 2, \ldots, i_0, N - i'_0, N - i'_0 + 1, \ldots, N - 1\,, \\ &\qquad\qquad r_i > 0\,,\quad i = i_0 + 1, N - i'_0 - 1\,, \\ &\text{or}\ \ i_0 = i'_0 = N - 1\,,\quad r_i = 0\ \forall\, i\,. \end{aligned} \tag{91b}$$

The last subcase is of R–symmetry scalars. It is also the only formula in the case under consideration — **ac** — valid for $N = 1$ (where there are no counterterms).

For $N > 1$ and $i_0 = i'_0 = 0$ formula (91) is equivalent to (90) with $\beta = \alpha_{15}$, $\beta' = \alpha_{3,4+N}$. Also holds the following decomposition:

$$\Big(\hat{L}_{\mathrm{long}}\Big)_{|d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}} \oplus \hat{L}_{\Lambda+\alpha_{15}+\alpha_{3,4+N}}\,,\quad r_1 r_{N-1} > 0\,, \tag{92}$$

$\hat{L}_{\mathrm{long}}$ being a long superfield with the same values of r_i as Λ and with $j_1, j_2 \geq N/2$.

• For $j_1 > 0$, $j_2 = 0$ the corresponding character formulae are combinations of

(82) and the conjugate to (80) [1]:

$$\mathrm{ch}\,\hat L_\Lambda = \sum_{\hat s\in \hat W^{a'c}_{i'_0}} (-1)^{\ell(\hat s)}\,\hat s\cdot \mathrm{ch}\,\hat V^\Lambda - \mathcal{R} = \tag{93a}$$

$$= \sum_{\hat s\in \hat W^{a'c}_{i'_0}} (-1)^{\ell(\hat s)}\,\hat s\cdot \Big(\mathrm{ch}\,\hat V^\Lambda - \mathcal{R}_{\rm long}\Big)\,, \quad r_1>0\,, \tag{93b}$$

$$\hat W^{a'c}_{i'_0} \equiv \hat W_{\beta_{12}} \times \hat W^{c}_{i'_0}\,, \quad \beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}\,. \tag{93c}$$

For $i_0 = i'_0 = 0$ holds the decomposition:

$$\Big(\hat L_{\rm long}\Big)_{|d=d^{ac}} = \hat L_\Lambda \oplus \hat L_{\Lambda+\alpha_{15}} \oplus \hat L_{\Lambda+\beta_{12}} \oplus \hat L_{\Lambda+\alpha_{15}+\beta_{12}}\,, \quad r_1 r_{N-1} > 0\,, \tag{94}$$

where $\hat L_{\rm long}$ is a long superfield with the same values of $j_2 (=0)$, r_i as Λ and with $j_1 \geq N/2$. Note that the UIR $\hat L_{\Lambda+\alpha_{15}}$ is also of the type **ac** under consideration, while the last two UIRs are short from type **bc** considered below.

For R–symmetry scalars we combine (84) and the conjugate to (80a):

$$\mathrm{ch}\,\hat L_\Lambda = \sum_{k=1}^N \prod_{i=1}^k e(\alpha_{4,5+N-i}) + \prod_{\substack{\alpha\in\Delta^+_{\bar 1} \\ \alpha\neq\alpha_{1,4+k}\,, \\ k=1,\dots,N \\ \varepsilon_2>0}} (1+e(\alpha)) - \mathcal{R}\,, \quad r_i = 0\ \forall\, i\,. \tag{95}$$

• The case $j_1 = 0$, $j_2 > 0$ is obtained from the previous one by conjugation. Here for $i_0 = i'_0 = 0$ holds the decomposition:

$$\Big(\hat L_{\rm long}\Big)_{|d=d^{ac}} = \hat L_\Lambda \oplus \hat L_{\Lambda+\alpha_{3,4+N}} \oplus \hat L_{\Lambda+\beta_{34}} \oplus \hat L_{\Lambda+\alpha_{3,4+N}+\beta_{34}}\,, \quad r_1 r_{N-1} > 0\,, \tag{96}$$

where $\hat L_{\rm long}$ is a long superfield with the same values of $j_1 (=0)$, r_i as Λ and with $j_2 \geq N/2$. Note that the UIR $\hat L_{\Lambda+\alpha_{3,4+N}}$ is again of the type **ac** under consideration, while the last two UIRs are actually from type **ad** considered below.

• For $j_1 = j_2 = 0$ the corresponding character formulae are combinations of (82) and its conjugate:

$$\mathrm{ch}\,\hat L_\Lambda = \sum_{\hat s\in \hat W^{a'c'}_{i'_0}} (-1)^{\ell(\hat s)}\,\hat s\cdot \Big(\mathrm{ch}\,\hat V^\Lambda - \mathcal{R}_{\rm long}\Big)\,, \quad r_1 r_{N-1}>0\,, \tag{97a}$$

$$\hat W^{a'c'}_{i'_0} \equiv \hat W_{\beta_{12}} \times \hat W_{\beta_{34}}\,. \tag{97b}$$

For $i_0 = i'_0 = 0$ holds the decomposition:

$$\Big(\hat L_{\rm long}\Big)_{|d=d^{ac}} = \hat L_\Lambda \oplus \hat L_{\Lambda+\beta_{12}} \oplus \hat L_{\Lambda+\beta_{34}} \oplus \hat L_{\Lambda+\beta_{12}+\beta_{34}}\,, \quad r_1 r_{N-1} > 0\,, \tag{98}$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of $j_1(=0)$, $j_2(=0)$, r_i as Λ. Note that the UIR $\hat{L}_{\Lambda+\beta_{12}}$ is of the type **bc**, $\hat{L}_{\Lambda+\beta_{34}}$ is of the type **ad**, $\hat{L}_{\Lambda+\beta_{12}+\beta_{34}}$ is of the type **bd**, these three being considered below.

For R–symmetry scalars we combine (84) and its conjugate:

$$\begin{aligned}\mathrm{ch}\, \hat{L}_\Lambda = & \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \\ & + \prod_{\substack{\alpha\in\Delta_1^+ \\ \varepsilon_1+\varepsilon_2>0 \\ \varepsilon_3+\varepsilon_4>0}} (1+e(\alpha)) - \mathcal{R}\,, \quad r_i = 0\ \forall\, i\,. \end{aligned} \tag{99}$$

•ad $\quad d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1$, $j_1 = 0$.
In these short DRC cases hold the three null conditions (67), and the conjugates to (85) and (86). In addition, for $N > 1$ if $r_i = 0$, $i = 1, \ldots, i_0$, hold (78) and if $r_{N-i} = 0$, $i = 1, \ldots, i'_0$, hold the conjugate of (87).

If $i_0 + i'_0 \leq N-2$ all eliminations are as described separately for cases **•a** and **•d**. All these are Grassmann–analytic UIRs. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-3-i_0-2i'_0}$. Interesting subcases are the so-called BPS states, cf., [14,17,20–25]. They are characterized by the number κ of odd generators which annihilate them — then the corresponding case is called $\frac{\kappa}{4N}$–BPS state. For example consider $N = 4$ and $\frac{1}{4}$–BPS cases with $z = 0 \Rightarrow d = 2m/N$. One such case is obtained for $i_0 = 1$, $i'_0 = 0$, $j_2 > 0$, then $d = \frac{1}{2}(2r_2 + 3r_3)$, $r_1 = 0$, $r_2 > 0$, $r_3 = 2(1 + j_2)$.

For $j_2 m_1 > 0$ the corresponding character formula is a combination of (80) and the conjugate of (88):

$$\mathrm{ch}\, \hat{L}_\Lambda = \sum_{\hat{s}\in \hat{W}^{ad}_{i_0,i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \mathrm{ch}\, \hat{V}^\Lambda - \mathcal{R}\,, \tag{100a}$$

$$\begin{aligned}\hat{W}^{ad}_{i_0,i'_0} \equiv \hat{W}^{a}_{i_0} \times \hat{W}^{d}_{i'_0}\,, \quad & j_2 m_1 > 0\,, \\ & r_i = 0\,, \quad i = 1,2,\ldots,i_0, N-i'_0, N-i'_0+1, \ldots, N-1\,, \\ & r_i > 0\,, \quad i = i_0+1, N-i'_0-1\,,. \end{aligned} \tag{100b}$$

For $i_0 = i'_0 = 0$ some of these UIRs appear (up to two times) in the decomposition (96) [1].

For $j_2 = 0$, $m_1 > 0$ the corresponding character formula is a combination of (82) and the conjugate of (88a):

$$\mathrm{ch}\, \hat{L}_\Lambda = \sum_{\hat{s}\in \hat{W}^{a'd}_{i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \mathrm{ch}\, \hat{V}^\Lambda - \mathcal{R}\,, \tag{101a}$$

$$\hat{W}^{a'd}_{i'_0} \equiv \hat{W}_{\beta_{12}} \times \hat{W}^{d}_{i'_0}\,, \tag{101b}$$

where $\beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}$. For $i_0 = i'_0 = 0$ some of these UIRs appear in the decomposition (98) or (96) [1].

In the case of R–symmetry scalars we have $i_0 = i'_0 = N-1$, $\kappa = 3N$ and all generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $X^+_{3,4+k}$ are eliminated. Here holds $d = -z = 1+j_2$. These anti-chiral irreps form one of the three series of **massless** UIRs; they are denoted χ^+_s, $s = j_2 = 0, \frac{1}{2}, 1, \ldots$, in Section 3 of [3]. Besides the vacuum they contain only N states in $\hat{L}_\Lambda$ given by (66) for $k = 1, \ldots, N$. These should be called ultrashort UIRs. The character formula can be written most explicitly:

$$\mathrm{ch}\,\hat{L}_\Lambda = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i})\,, \quad j_1 = r_i = 0\ \forall\, i\,, \tag{102}$$

and it is valid for any j_2. In the case under consideration — **ad** — only the last character formula is valid for $N=1$.

•bc $\quad d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1$, $j_2 = 0$, $z = 2m/N - m_1 + 1 + j_1$.

This case is conjugate to the previous one and all results may be obtained by the substitutions given for the SRC conjugate cases.

•bd $\quad d = d^2_{N1} = d^4_{NN} = m_1$, $j_1 = j_2 = 0$, $z = 2m/N - m_1$.

In these short DRC cases hold the four null conditions (85), (86), and their conjugates.

For $N = 1$ this is the trivial irrep with $d = z = 0$. This follows from the fact that since $d = j_1 = j_2 = 0$ also holds the even reducibility condition (16b) (and consequently (16d, e, f)). Thus, we have the null conditions: $X^+_k|\Lambda\rangle = 0$ for all simple root generators (and consequently for all generators) and the irrep consists only of the vacuum $|\Lambda\rangle$.

For $N > 1$ the situation is non-trivial. In addition to the mentioned conditions, and if $r_i = 0$, $i = 1, \ldots, i_0$, hold (87) and if $r_{N-i} = 0$, $i = 1, \ldots, i'_0$, hold the conjugates of (87).

If $i_0 + i'_0 \leq N-2$ all eliminations are as described separately for cases **•b** and **•d**. These are also Grassmann–analytic UIRs. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-4-2i_0-2i'_0}$. For $N = 4$ for the BPS cases we take $z = \frac{1}{2}(r_3 - r_1) = 0 \Rightarrow d = 2r_1 + r_2$. In the $\frac{1}{4}$–BPS case we have $i_0 = i'_0 = 0$, $r_1 = r_3 > 0$.

For $i_0 = i'_0 = 0$ some of these UIRs appear in the decomposition (98) [1].

Most interesting is the case $i_0 + i'_0 = N-2$, then there is only one non-zero r_i, namely, $r_{1+i_0} = r_{N-1-i'_0} > 0$, while the rest r_i are zero. Thus, the Young tableau parameters are: $m_1 = r_{1+i_0}$, $m = (1+i_0)r_{1+i_0}$.

An important subcase is when $d = m_1 = 1$, then $m = i_0 + 1 = N - 1 - i'_0$, $r_i = \delta_{mi}$, and these irreps form the third series of **massless** UIRs. In Section 3 of [3] they are denoted χ'_n, $n = m \geq \frac{1}{2}N$, $(z = 2n/N - 1)$, χ'^+_n, $n = N - m \geq \frac{1}{2}N$, $(z = 1 - 2n/N)$. Note that for even N there is the coincidence: $\chi'_n = \chi'^+_n$, where $n = m = N - m = N/2$. Here we shall parametrize these UIRs by the parameter $i_0 = 0, 1, \ldots, N-1$.

Another subcase here are $\frac{1}{2}$–BPS states for even N with $z = 0 \Rightarrow d = m_1 = 2m/N \Rightarrow i_0 = i'_0 = N/2 - 1 \Rightarrow m_1 = r_{N/2}$, $m = \frac{N}{2}\, r_{N/2}$. These are also massless only if $r_{N/2} = 1$, which is the self-conjugate case: χ'_n, $n = N/2$. For $N = 4$ we have: $i_0 = i'_0 = 1$, $r_1 = r_3 = 0$, $r_2 > 0$, which is also massless if $r_2 = 1$.

Finally, in the case of R–symmetry scalars we have $i_0 = i'_0 = N-1$ and all $4N$ odd generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $X^+_{3,4+k}$, $X^+_{4,4+k}$, are eliminated. More than this, all quantum numbers are zero, (cf. (39d)), and this is the trivial irrep. The latter follows exactly as explained above for the case $N=1$.

For $m_1 > 0$ the corresponding character formula is a combination of (88) and its conjugate:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^{bd}_{i_0,i'_0}} (-1)^{\ell(\hat{s})}\hat{s}\cdot\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}\,, \tag{103a}$$

$$\begin{aligned} &\hat{W}^{bd}_{i_0,i'_0} \equiv \hat{W}^b_{i_0}\times\hat{W}^d_{i'_0}\,, \\ &\qquad r_i = 0\,, \quad i = 1,2,\ldots,i_0, N-i'_0, N-i'_0+1,\ldots,N-1\,, \\ &\qquad r_i > 0\,, \quad i = i_0+1, N-i'_0-1\,, \end{aligned} \tag{103b}$$

where $\mathcal{R}$ designates the counter–terms due to our Criterion, in particular, due to (58) taken with $\varepsilon_{a,N+1-k} = 0$, $a = 1,2$, $k = 1,\ldots,1+i'_0$, $\varepsilon_{bj} = 0$, $b = 3,4$, $k = j,\ldots,1+i_0$.

Also for the third series of massless UIRs we can give a much more explicit character formula without counter–terms. Fix the parameter $i_0 = 0,1,\ldots,N-2$. Then there are only the following states in $\hat{L}_\Lambda$:

$$X^+_{2,N+4-j}\cdots X^+_{2,N+4-i_0}|\Lambda\rangle\,, \quad j = 0,1,\ldots,i_0\,, \tag{104a}$$

$$X^+_{4,4+k}\cdots X^+_{4,N+3-i_0}|\Lambda\rangle\,, \quad k = 1,\ldots,N-1-i_0\,, \tag{104b}$$

altogether N states besides the vacuum [1]. The corresponding character formula for the massless UIRs of this series is therefore:

$$\begin{aligned} \mathrm{ch}\,\hat{L}_\Lambda &= 1 + \sum_{j=0}^{i_0}\prod_{i=j}^{i_0} e(\alpha_{2,N+4-i}) + \sum_{k=1}^{N-1-i_0}\prod_{i=k}^{N-1-i_0} e(\alpha_{4,4+i})\,, \\ &\qquad i_0 = 0,1,\ldots,N-2\,, \quad r_i = \delta_{i,i_0+1}\,. \end{aligned} \tag{105}$$

4 Discussion and outlook

First we recall the results on decompositions of long irreps as they descend to the unitarity threshold. In the SRC cases in subsection (3.3) we have established that for $d = d_{\max}$ there hold the two–term decompositions given in (77), (83), and their conjugates. In the DRC cases in subsection (3.4) we have established that for $N>1$ and $d = d_{\max} = d^{ac}$ hold the four–term decompositions given in (92), (94), (96), (98).

Next we note that for $N=1$ all SRC cases enter some decomposition, while no DRC cases enter any decomposition. For $N>1$ the situation is more diverse and so we give the list of UIRs that do **not** enter decompositions:

- **SRC cases:**

•a $\quad d = d_{\max} = d^a = d^1_{N1} = 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN}$,
j_1, j_2 arbitrary, $r_1 = 0$.

•b $d = d^2_{N1} = z + 2m_1 - 2m/N > d^3_{NN}$, $j_2 = 0$,
j_1 arbitrary, $r_1 \leq 2$.
•c $d = d_{\max} = d^c = d^3_{NN} = 2 + 2j_1 - z + 2m/N > d^1_{N1}$,
j_1, j_2 arbitrary, $r_{N-1} = 0$.
•d $d = d^4_{NN} = -z + 2m/N > d^1_{N1}$, $j_1 = 0$,
j_2 arbitrary, $r_{N-1} \leq 2$.
• **DRC cases:** all non-trivial cases for $N = 1$, while for $N > 1$ the list is:
•ac $d = d_{\max} = d^{ac} = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2 + m_1$,
j_1, j_2 arbitrary, $r_1 r_{N-1} = 0$.
•ad $d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1$, $j_1 = 0$,
j_2 arbitrary, $r_{N-1} \leq 2$, $r_1 = 0$ for $N > 2$.
•bc $d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1$, $j_2 = 0$,
j_1 arbitrary, $r_1 \leq 2$, $r_{N-1} = 0$ for $N > 2$.
•bd $d = d^2_{N1} = d^4_{NN} = m_1$, $j_1 = j_2 = 0$,
$r_1, r_{N-1} \leq 2$ for $N > 2$, $r_1 \leq 4$ for $N = 2$.

We would like to point out possible application of our results to current developments in conformal field theory. Recently, there is interest in superfields with conformal dimensions which are protected from renormalisation in the sense that they cannot develop anomalous dimensions [14–16, 18, 26, 27]. Initially, the idea was that this happens because the representations under which they transform determine these dimensions uniquely. Later, it was argued that one can tell which operators will be protected in the quantum theory simply by looking at the representations they transform under and whether they can be written in terms of single trace 1/2 BPS operators (chiral primaries or CPOs) on analytic superspace [18]. In [27] it was shown how, at the unitarity threshold, a long multiplet can be decomposed into four semi–short multiplets, and decompositions similar to ours, i.e., involving the modules given in (90) and [1] (which follow from the odd embeddings given in [7]), were considered for $N = 2$, 4. However, the decompositions of [27] are justified on the dimensions of the finite–dimensional irreps of the Lorentz and $su(N)$ subalgebras involved in the superfields involved in the decompositions, and in particular, the latter hold also when $r_1 r_{N-1} = 0$.

Independently of the above, we would like to make a *mathematical* remark. As a by-product of our analysis we have obtained character formulae for the complex Lie superalgebras $sl(4/N)$. The point is that our character formulae have as starting point character formulae of Verma modules and factor–modules over $sl(4/N)$. Thus, almost all character formulae in Section 3, more precisely, formulae (62), (68), (71), (75), (80), (82), (88), (90), (91), (93), (97), (100), (101), (102), (103) and their conjugates become character formulae for $sl(4/N)$ for the same values of the representation parameters by just discarding the counter-terms $\mathcal{R}$, $\mathcal{R}_{\rm long}$, resp.

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