Solutions of the Camassa−Holm hierarchy in 2+1 dimensions

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We consider solutions of a generalization of the Camassa–Holm hierarchy to (2+1) dimensions that include, in particular, the well-known multipeakons solutions for the celebrated Camassa–Holm equation.

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1 Introduction

Camassa and Holm (1993) found a completely integrable dispersive shallow–water equation, namely

$$u_t + 2ku_x - u_{xxt} + 3uu_x - 2u_x u_{xx} + uu_{xxx} = 0\,, \tag{1}$$

where u is the fluid velocity in the x direction, and k a constant related to the critical shallow–water wave speed [5]. The limit $k = 0$, that is

$$u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} = -\left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}\right)_x\,, \tag{2}$$

was given special attention in [6] because of its mathematical interest; seeking solutions of the form $u(x,t) = U(x-ct)$ (U a function that vanishes at infinity with its first and second derivatives) it was found that

$$U = c\mathrm{e}^{-|x-ct|} + O\left(k \log k\right) \tag{3}$$

and, therefore, for $k = 0$, the solution (3) was seen [6] as the composition of the two exponentials that satisfy the equation

$$\left(U' - c\right)\left[(U')^2 - U^2\right] = 0\,. \tag{4}$$

The form of the travelling wave solution (3) led Camassa and Holm to make the well-known solution ansatz for N interacting peaked solutions for the CH (Camassa–Holm) equation (2) (given in detail in section (2.2)).

From then onwards, a lot of attention was given to peakons solutions; for instance, in 1994, Alber, Camassa, Holm and Marsden [1] investigated the geometry of peaked solitons for the general CH equation, that is equation (1), without assuming $k = 0$. The existence of a Liouville transformation mapping CH spectral problem to the string problem was used by Beals, Sattinger and Szmigielski [2], [3], in 1999, to present a close form of the multi–peakons solutions using a theorem of Stieltjes on continuous fractions. The same authors [4] investigated the relationship between the multipeakons and the classical moment problem. In 2000, Constantin and Strauss [8] studied the stability of peakons and Lenells [15] studied the stability of periodic peakons and, in [16], presented a variational proof of it.

Later, in 2003, an integrable equation with peakons solutions was investigated by Degasperis, Holm and Hone [9]; this equation, as the authors say, is of a similar form to CH shallow water equation and was obtained by the method of asymptotic integrability. In its dispersionless form it was written

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \tag{5}$$

and proved that the single peakon

$$u(x,t) = c\mathrm{e}^{-|x-ct|} \tag{6}$$

was a solution. The authors considered too the family of equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} \tag{7}$$

for real parameter b [12], including both CH ($b = 2$) and (5) ($b = 3$) as special cases. It turned out that all equations of the family have multi–peakons solutions and satisfy the dynamical system

$$\begin{aligned} \dot{p}_j &= -(b-1)\frac{\partial G}{\partial q_j}\,, \\ \dot{q}_j &= \frac{\partial G}{\partial p_j}\,, \end{aligned} \tag{8}$$

where

$$G = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k \mathrm{e}^{-|q_j - q_k|} \tag{9}$$

taking the Hamiltonian form in the case $b = 2$ only. Hone and Wang [14] isolated the peakons equations via the Wahlquist–Estabrook prolongation algebra method.

In [13], it was shown that the CH spectral problem gives two different integrable hierarchies of nonlinear evolutions equations: one is of negative order CH hierarchy while the other one is of positive order CH hierarchy. Besides it was seen that the celebrated CH equation is included in the negative order CH hierarchy while the Dym type equation is included in the positive order CH hierarchy. Many papers deal with solutions of the CH hierarchy, for instance [11].

In section 2 of this paper we consider a generalization [7], [10] of the CH hierarchy to (2+1) dimensions. Following the lead due to Camassa and Holm we make an ansatz about the existence of certain multipeakons solutions in the different equations of the hierarchy and present the resulting dynamical system. This is, naturally, a PDE system and so the equations appearing are of different type; there is a set of equations involving derivatives with respect to the variable y, another one which give the recursion relations and finally the evolution equations. In section 3, examples of the dynamical systems in some particular cases are presented.

Section 4 deals with the reduction of the hierarchy to (1+1) dimensions; the dynamical system is explicitly given as well as some examples.

Conclusions are listed in the pertinent section and two appendices added with formulae needed in the computations.

2 The negative Camassa−Holm hierarchy NCCH in 2+1 dimensions

Consider the well-known negative Camassa–Holm hierarchy NCHH [13], for a field $u(x,t)$, that is

$$u_t = R^{-n} u_{x,} \, , \quad R = J_0 J_1^{-1} \, , \quad n \geq 1 \, , \tag{10}$$

where n, the order of the hierarchy, is an integer number and J_0, J_1 are the following operators

$$J_0 = \partial^3 - \partial \, , \quad J_1 = u\partial + \partial u \, , \quad \partial = \frac{\partial}{\partial x} \, . \tag{11}$$

Introducing n functions $v_1(x,t)$, $\ldots$, $v_n(x,t)$ as in [10]

$$\begin{aligned} v_1 &= J_0^{-1} u_x \Rightarrow J_0 v_1 = u_x \, , \\ v_k &= J_0^{-1} J_1 v_{k-1} \Rightarrow J_0 v_{kj} = J_1 v_{k-1} \, , \quad k = 2, \ldots, n \, , \end{aligned} \tag{12}$$

the equation (10) can be written as

$$u_t = J_1 v_n \tag{13}$$

and therefore the negative Camassa–Holm hierarchy can be considered as the $n+1$ equations (12), (13) in $n+1$ fields u, v_1, $\ldots$, v_n.

Obviously for $n=1$, the system (12), (13) reduces to

$$\begin{aligned} u_t &= 2u(v_1)_x + u_x v_1 \, , \\ u &= (v_1)_{xx} - v_1 \, , \end{aligned} \tag{14}$$

which is the well-known Camassa–Holm equation [5].

2.1 Generalization to 3 dimensions

As it was shown in [10] a simple generalization of (12), (13) to 3 dimensions is the following system

$$\begin{aligned} U_y &= J_0 V_1 \, , \\ J_0 V_k &= J_1 V_{k-1} \, , \quad k = 2, \ldots, n \, , \\ U_t &= J_1 V_n \, , \end{aligned} \tag{15}$$

where $U = U(x,t,y)$ and $V_j = V_j(x,t,y)$, that can be written as

$$U_t = R^{-n} U_y \tag{16}$$

and denoted by CHH(2+1). It is trivial to see that the negative Camassa–Holm hierarchy (10) is obtained from (16) by using the reduction $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$.

2.2 Peaked solitons for the Camassa−Holm equation CH(1+1)

In [6] it was made the following solution ansatz for N interacting peaked solutions for the CH equation (14)

$$v_1(x,t) = \sum_{i=1}^{N} p_i(t) \mathrm{e}^{-|x-q_i(t)|}, \tag{17}$$

where

$$u(x,t) = 2\sum_{i=1}^{N} p_j(t)\delta\big[x - q_j(t)\big] \tag{18}$$

and the peaks in v_1 are delta functions in u. Substituting (17) and (18) in (14) yields the evolution equations for q_j and p_j

$$\begin{aligned} \dot{q} &= \sum_{i=1}^{N} p_j \mathrm{e}^{-|q_i-q_j|}\,, \\ \dot{p} &= p_i \sum_{i=1}^{N} p_j \,\mathrm{sgn}(q_i - q_j)\mathrm{e}^{-|q_i-q_j|}\,. \end{aligned} \tag{19}$$

These equations are Hamilton's canonical equations, with Hamiltonian H_A, given by

$$H_A = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j \mathrm{e}^{-|q_i-q_j|}\,. \tag{20}$$

2.3 Solutions of the negative Camassa-Holm hierarchy NCCH (2+1)

We make, now, the following ansatz for the negative Camassa–Holm hierarchy NCCH(2+1)

$$V_k(x,y,t) = \sum_{i=1}^{N} A_i^{k)}(y,t)\partial^{-1}\left(\mathrm{e}^{-|x-q_i(y,t)|}\right) + \sum_{i=1}^{N} B_i^{k)}(y,t)\mathrm{e}^{-|x-q_i(y,t)|}, \tag{21}$$

$k = 1,2,\ldots,n$, where $A_i^{1)}(y,t) = 0$, $\forall i = 1,2,\ldots,N$, that is

$$V_1(x,y,t) = \sum_{i=1}^{N} B_i^{1)}(y,t)\mathrm{e}^{-|x-q_i(y,t)|} \tag{22}$$

is a multipeakon solution and, according to (15), U is:

$$U(x,y,t) = -2\sum_{i=1}^{N}\gamma_i(y,t)\delta\big(x-q_i(y,t)\big)\,. \tag{23}$$

So that, as in the CH equation, the peaks in V_1 are delta functions in U.

Substituting (21) and (23) into (15) and using the identities of the two appendices yields the explicit formulation of the resulting system. We remark the three different sets of equations that appear therein; first, $2N$ equations (24) involving derivatives with respect to the variable y, second, the recursion relations (25) and finally the evolution equations (26)

$$\begin{aligned}\frac{\partial}{\partial y}(\gamma_i(y,t)) &= 0\,,\\ \gamma_i(y,t)\frac{\partial}{\partial y}\big(q_i(y,t)\big) &= -B_i^{1)}(y,t)\,,\quad i=1,2,\ldots,N\,,\end{aligned} \tag{24}$$

$$\begin{aligned}A_i^{k)}(y,t) &= \sum_{j=1}^{N}\gamma_i(y,t)A_j^{k-1)}(y,t)\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|} -\\ &\quad -\sum_{j=1}^{N}\gamma_i(y,t)B_j^{k-1)}(y,t)\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|}\,\mathrm{sgn}(q_i(y,t)-q_j(y,t))\,,\\ B_i^{k)}(y,t) &= \sum_{j=1}^{N}\gamma_i(y,t)A_j^{k-1)}(y,t)\left(\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|}-1\right)\mathrm{sgn}\big(q_i(y,t)-q_j(y,t)\big) +\\ &\quad +\sum_{j=1}^{N}\gamma_i(y,t)B_j^{k-1)}(y,t)\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|}\,,\\ k &= 2,\ldots,n\,,\quad i=1,2,\,\ldots,N\,.\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\gamma_i(y,t)) &= \gamma_i(y,t)\sum_{j=1}^{N}A_j^{n)}(y,t)\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|} -\\ &\quad -\gamma_i(y,t)\sum_{j=1}^{N}B_j^{n)}(y,t)\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|}\,\mathrm{sgn}\big(q_i(y,t)-q_j(y,t)\big)\,,\\ \frac{\partial}{\partial t}(q_i(y,t)) &= \sum_{j=1}^{N}A_j^{n)}(y,t)\left(\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|}-1\right)\mathrm{sgn}\big(q_i(y,t)-q_j(y,t)\big) -\\ &\quad -\sum_{j=1}^{N}B_j^{n)}(y,t)\mathrm{e}^{-|q_i(y,t)-q_j(y,t)|}\,,\\ i &= 1,2,\ldots,N\,.\end{aligned} \tag{26}$$

3 Examples in 2+1 dimensions

3.1 Case $n = 1$ and $N = 1$

We now consider the system

$$\begin{aligned} U_y &= (V_1)_{xxx} - (V_1)_x \,, \\ U_t &= 2U(V_1)_x + U_x V_1 \,, \end{aligned} \tag{27}$$

which corresponds to the choice $n = 1$ in the hierarchy (15). For $N = 1$, that is for

$$\begin{aligned} V_1(x,y,t) &= p(y,t)\mathrm{e}^{-|x-q(y,t)|}, \\ U(x,y,t) &= -2\gamma(y,t)\delta\big(x - q(y,t)\big) \end{aligned} \tag{28}$$

it comes out that

$$\begin{aligned} \frac{\partial}{\partial y}\big(\gamma(y,t)\big) &= 0\,, \\ \frac{\partial}{\partial y}\big(q(y,t)\big) &= -\frac{p(y,t)}{\gamma(y,t)}\,, \\ \frac{\partial}{\partial t}\big(\gamma(y,t)\big) &= 0\,, \\ \frac{\partial}{\partial t}\big(q(y,t)\big) &= -p(y,t)\,, \end{aligned} \tag{29}$$

which implies

$$\begin{aligned} \gamma(y,t) &= \gamma_0\,, \\ \frac{\partial}{\partial t}q(y,t) &= \gamma_0\left(\frac{\partial}{\partial y}q(y,t)\right) \end{aligned} \tag{30}$$

that is $q(y,t) = F(y + \gamma_0 t)$ and so

$$\begin{aligned} V_1(x,y,t) &= -\left(\frac{\partial}{\partial t}q(y,t)\right)\mathrm{e}^{-|x-q(y,t)|}, \\ U(x,y,t) &= -2\gamma_0\delta\big(x - q(y,t)\big)\,. \end{aligned} \tag{31}$$

Note that assuming that $q(y,t)$ is of the form $q(y,t) = y + \gamma_0 t$, then a peakon solution of the previous system is given by

$$\begin{aligned} V_1(x,y,t) &= \gamma_0\mathrm{e}^{-|x-y-\gamma_0 t|}\,, \\ U(x,y,t) &= -2\gamma_0\delta(x - y - \gamma_0 t)\,. \end{aligned} \tag{32}$$

3.2 Case $n = 1$ and $N = 2$. Two−soliton dynamics

Consider the system (27) and take

$$\begin{aligned} V_1(x,y,t) &= p_1(y,t)\mathrm{e}^{-|x-q_1(y,t)|} + p_2(y,t)\mathrm{e}^{-|x-q_2(y,t)|} \\ U(x,y,t) &= -2\gamma_1(y,t)\delta\big(x - q_1(y,t)\big) - 2\gamma_2(y,t)\delta\big(x - q_2(y,t)\big)\,. \end{aligned} \tag{33}$$

Substitution of (33) in (27) gives the dynamical system for the two–soliton case, explicitly

$$
\begin{aligned}
&\frac{\partial}{\partial y}(\gamma_1(y,t)) = \frac{\partial}{\partial y}(\gamma_2(y,t)) = 0\,,\\
&\frac{\partial}{\partial y}(q_1(y,t)) = -\frac{p_1(y,t)}{\gamma_1(y,t)}\,,\\
&\frac{\partial}{\partial y}(q_2(y,t)) = -\frac{p_2(y,t)}{\gamma_2(y,t)}\,,\\
&\frac{\partial}{\partial t}(\gamma_1(y,t)) = -\gamma_1(y,t)p_2(y,t)\mathrm{e}^{-|q_1(y,t)-q_2(y,t)|}\,\mathrm{sgn}(q_1(y,t)-q_2(y,t))\,,\\
&\frac{\partial}{\partial t}(\gamma_2(y,t)) = -\gamma_2(y,t)p_1(y,t)\mathrm{e}^{-|q_2(y,t)-q_1(y,t)|}\,\mathrm{sgn}(q_2(y,t)-q_1(y,t))\,,\\
&\frac{\partial}{\partial t}(q_1(y,t)) = -p_1 - p_2\mathrm{e}^{-|q_1(y,t)-q_2(y,t)|}\,,\\
&\frac{\partial}{\partial t}(q_2(y,t)) = -p_1\mathrm{e}^{-|q_2(y,t)-q_1(y,t)|} - p_2(y,t)\,.
\end{aligned}
\tag{34}
$$

Note that γ_1 and γ_2 are independent of y and calling

$$F(t) = p_2(y,t)\mathrm{e}^{|q_1(y,t)-q_2(y,t)|}\,\mathrm{sgn}(q_1(y,t)-q_2(y,t)) \tag{35}$$

and

$$G(t) = p_1(y,t)\mathrm{e}^{|q_2(y,t)-q_1(y,t)|}\,\mathrm{sgn}(q_2(y,t)-q_1(y,t))\,, \tag{36}$$

it follows

$$
\begin{aligned}
\ln\gamma_1(t) &= -\int F(t)\mathrm{d}t\,,\\
\ln\gamma_2(t) &= -\int G(t)\mathrm{d}t
\end{aligned}
\tag{37}
$$

and so

$$U(y,t) = -2\mathrm{e}^{-\int F(t)\mathrm{d}t}\delta\big(x-q_1(y,t)\big) - 2\mathrm{e}^{-\int G(t)\mathrm{d}t}\delta\big(x-q_2(y,t)\big)\,, \tag{38}$$

where

$$
\begin{aligned}
&\frac{\partial}{\partial y}(q_1(y,t)) = -\mathrm{e}^{\int F(t)\mathrm{d}t}p_1(y,t)\,,\\
&\frac{\partial}{\partial y}(q_2(y,t)) = -\mathrm{e}^{\int G(t)\mathrm{d}t}p_2(y,t)\,,\\
&\frac{\partial}{\partial t}(q_1(y,t)) = -p_1(y,t) - p_2(y,t)\mathrm{e}^{-|q_1(y,t)-q_2(y,t)|},\\
&\frac{\partial}{\partial t}(q_2(y,t)) = -p_1(y,t)\mathrm{e}^{-|q_2(y,t)-q_1(y,t)|} - p_2(y,t)\,.
\end{aligned}
\tag{39}
$$

3.3 Case $n = 2$ and $N = 1$

For the system

$$
\begin{aligned}
U_y &= (V_1)_{xxx} - (V_1)_x\,,\\
(V_2)_{xxx} - (V_2)_x &= 2U(V_1)_x + U_xV_1\,,\\
U_t &= 2U(V_2)_x + U_xV_2
\end{aligned}
\tag{40}
$$

taking V_1, U as in (28) and

$$V_2(x,y,t) = A_1^{2)}(y,t)\partial^{-1}\left(\mathrm{e}^{-|x-q(y,t)|}\right) + B_1^{2)}(y,t)\mathrm{e}^{-|x-q(y,t)|}\,, \tag{41}$$

we get

$$\begin{aligned}
\frac{\partial}{\partial y}(\gamma(y,t)) &= 0\,,\\
\frac{\partial}{\partial y}(q(y,t)) &= -\frac{p(y,t)}{\gamma(y,t)}\,,\\
A_1^{2)}(y,t) &= 0\,,\\
B_1^{2)}(y,t) &= \gamma(y,t)p(y,t)\,,\\
\frac{\partial}{\partial t}(\gamma(y,t)) &= 0\,,\\
\frac{\partial}{\partial t}(q(y,t)) &= -\gamma(y,t)p(y,t)\,.,
\end{aligned} \tag{42}$$

and, consequently,

$$\begin{aligned}
\gamma(y,t) &= \text{constant} = \gamma_0\,,\\
V_1(x,y,t) &= -\gamma_0\left(\frac{\partial}{\partial y}\,q(y,t)\right)\mathrm{e}^{-|x-q(y,t)|},\\
V_2(x,y,t) &= -\left(\frac{\partial}{\partial t}\,q(y,t)\right)\mathrm{e}^{-|x-q(y,t)|},\\
U(x,y,t) &= -2\gamma_0\delta\big(x-q(y,t)\big)\,,\\
\frac{\partial}{\partial t}\,,q(y,t) &= \gamma_0^2\left(\frac{\partial}{\partial y}\,q(y,t)\right),
\end{aligned} \tag{43}$$

where $q(y,t) = F\big(y+\gamma_0^2 t\big)$.

Note that in this case, assuming $q(y,t) = y + \gamma_0^2 t$, a solution of the system (43) is

$$\begin{aligned}
V_1(x,y,t) &= -\gamma_0 a\mathrm{e}^{-\left|x-y-\gamma_0^2 t\right|},\\
V_2(x,y,t) &= -\gamma_0^2\mathrm{e}^{-\left|x-y-\gamma_0^2 t\right|},\\
U(x,y,t) &= -2\gamma_0\delta\big(x-y-\gamma_0^2 t\big)
\end{aligned} \tag{44}$$

and, therefore, both variables V_1 and V_2 have peakon solutions.

4 Reduction to two dimensions

Let us consider the reduction $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$ of the system (15) which gives us the negative CH hierarchy in (1+1) dimensions. Taking $\frac{\partial}{\partial t} = 0$, we get the positive hierarchy, where t is relabelled as y. We are only interested in the negative hierarchy as the other one does not have peakons as solutions.

The resulting dynamical system, in this case, has only two sets of equations, the recursion relations (45) and the evolution equations (46). Explicitly:

$$\begin{aligned} A_i^{k)}(y,t) &= \sum_{j=1}^N B_i^{1)}(t) A_j^{k-1)}(t) \mathrm{e}^{-|q_i(t)-q_j(t)|} - \\ &\quad - \sum_{j=1}^N B_i^{1)}(t) B_j^{k-1)}(t) \mathrm{e}^{-|q_i(t)-q_j(t)|} \operatorname{sgn}\big(q_i(t)-q_j(t)\big)\,, \\ B_i^{k)}(t) &= \sum_{j=1}^N B_i^{1)}(t) A_j^{k-1)}(t) \left(\mathrm{e}^{-|q_i(t)-q_j(t)|} - 1\right) \operatorname{sgn}\big(q_i(t)-q_j(t)\big) + \\ &\quad + \sum_{j=1}^N B_i^{1)}(t) B_j^{k-1)}(t) \mathrm{e}^{-|q_i(t)-q_j(t)|}\,, \\ k &= 2,\dots,n, \quad i = 1,2,\dots,N \end{aligned} \tag{45}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} B_i^{1)}(t) &= B_i^{1)}(t) \sum_{j=1}^N A_j^{n)}(t) \mathrm{e}^{-|q_i(t)-q_j(t)|} - \\ &\quad - B_i^{1)}(t) \sum_{j=1}^N B_j^{n)}(t) \mathrm{e}^{-|q_i(t)-q_j(t)|} \operatorname{sgn}\big(q_i(t)-q_j(t)\big)\,, \\ \frac{\partial}{\partial t} q_i(t) &= \sum_{j=1}^N A_j^{n)}(t) \left(\mathrm{e}^{-|q_i(t)-q_j(t)|} - 1\right) \operatorname{sgn}\big(q_i(t)-q_j(t)\big) - \\ &\quad - \sum_{j=1}^N B_j^{n)}(t) \mathrm{e}^{-|q_i(t)-q_j(t)|}, \\ i &= 1,2,\dots,N\,, \end{aligned} \tag{46}$$

where

$$\begin{aligned} V_k(x,t) &= \sum_{i=1}^N A_i^{k)}(t) \partial^{-1}\big(\mathrm{e}^{-|x-q_i(t)|}\big) + \sum_{i=1}^N B_i^{k)}(t) \mathrm{e}^{-|x-q_i(t)|}, \quad k = 1,2,\dots,n\,, \\ A_i^{1)}(t) &= 0\,, \quad i = 1,2,\dots,N \\ U(x,t) &= -2 \sum_{i=1}^N B_i^{1)}(t) \delta\big(x-q_i(t)\big)\,. \end{aligned} \tag{47}$$

4.1 Examples in (1+1) dimensions

Taking $n=1$ and any N we get the well-known dynamical system of ODE for the CH equation.

We write explicitly the resulting dynamical system for the case $n=2$ (second element of the hierarchy) and any N; this case illustrates completely the complexity obtained when the number n increases.

The second element of the hierarchy is

$$\begin{aligned} (V_1)_{xxx} - (V_1)_x &= U_x\,, \\ (V_2)_{xxx} - (V_2)_x &= 2U(V_1)_x + U_x V_1\,, \\ U_t &= 2U(V_2)_x + U_x V_2\,. \end{aligned} \tag{48}$$

Taking U, V_1 and V_2 as in (47) and substituting in (48) we get the evolution equations (where the coefficients $A_i^{2)}$ and $B_i^{2)}$ are computed using the recursion relations)

$$\begin{aligned} \frac{\partial}{\partial t}B_i^{1)}(t) &= -B_i^{1)}\left[\sum_{j,k=1}^{N} B_j^{1)}B_k^{1)}\mathrm{e}^{-|q_j-q_k|}\mathrm{e}^{-|q_i-q_j|}\big(\mathrm{sgn}(q_j-q_k)+\mathrm{sgn}(q_i-q_j)\big)\right], \\ \frac{\partial}{\partial t}q_i(t) &= -\sum_{j,k=1}^{N} B_j^{1)}B_k^{1)}\mathrm{e}^{-|q_j-q_k|}\Big[\big(\mathrm{e}^{-|q_i-q_j|}-1\big)\,\mathrm{sgn}(q_j-q_k)\,\mathrm{sgn}(q_i-q_k) + \\ &\qquad \mathrm{e}^{-|q_i-q_j|}\Big], \\ i &= 1,2,\ldots,N\,. \end{aligned} \tag{49}$$

5 Conclusions

Here we consider a generalization of the CH hierarchy to (2+1) dimensions and study certain solutions.

Our main conclusions are as follows:

a) Find the dynamical system, remarking the different sets of equations involved therein; that is, equations having derivatives with respect to the spatial variable y, the recursion relations and the evolution equations.

b) Analyze the resulting dynamical system in some particular cases proving the existence of peakons solutions.

c) Consider the reduction $\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$ of the CH hierarchy in (2+1) dimensions that gives the well-known CH hierarchy in (1+1) dimensions, pointing out, with an example, the complexity of the dynamical system as the number n (order of the hierarchy) increases. For $n = 1$, that is for the CH equation, the dynamical system is also very well-known.

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Appendix A

We list below some formulae that are needed for the calculations made in section 2. Let $\partial(x)$ be the Dirichlet distribution, that is $\partial(x)(f(x)) = f(0)$ and δ' its first derivative, that is $\delta'(f(x)) = -f'(0)$. Then, assuming that a and b are functions of y and t, $a = a(y,t)$, $b = b(y,t)$, the following are true:

$$\mathrm{e}^{-|x-a|}\partial(x-b) = \mathrm{e}^{-|b-a|}\partial(x-b)\,, \tag{50}$$

$$\mathrm{e}^{-|x-a|}\,\mathrm{sgn}(x-a)\partial(x-b) = \mathrm{e}^{-|b-a|}\,\mathrm{sgn}(b-a)\partial(x-b)\,, \tag{51}$$

$$\mathrm{e}^{-|x-a|}\delta'(x-b) = \mathrm{e}^{-|b-a|}\,\mathrm{sgn}(b-a)\delta(x-b) + \mathrm{e}^{-|b-a|}\delta'(x-b)\,, \tag{52}$$

$$\partial^{-1}\left(\mathrm{e}^{-|x-a|}\right) = \left(\mathrm{e}^{-|x-a|} - 1\right)\mathrm{sgn}(a-x)\,, \tag{53}$$

$$\partial^{-1}\left(\mathrm{e}^{-|x-a|}\right)\delta'(x-b) = -\mathrm{e}^{-|b-a|}\delta(x-b) - \left(\mathrm{e}^{-|b-a|} - 1\right)\mathrm{sgn}(b-a)\delta'(x-b)\,. \tag{54}$$

Appendix B

We specify, in this second appendix, the computations made to get the dynamical system for the NCCH(2+1). Take $\gamma_i = \gamma_i(y,t)$, $q_i = q_i(y,t)$, $B_i^{1)} = B_i^{1)}(y,t)$, $A_i^{k)} = A_i^{k)}(y,t)$, $B_i^{k)} = B_i^{k)}(y,t)$, $k = 2,\ldots,n$.

Equations (24) come from substituting in the first equation $U_y = J_0 V_1$ (independent of n), the following results

$$U_y = -2\sum_{i=1}^{N}\left(\frac{\partial}{\partial y}\gamma_i\right)\delta(x-q_i) + 2\sum_{i=1}^{N}\gamma_i\left(\frac{\partial}{\partial y}q_i\right)\delta'(x-q_i) \tag{55}$$

and

$$J_0V_1 = -2\sum_{i=1}^{N} B_i^{1)}\delta'(x-q_i)\,. \tag{56}$$

The recursion relations (25) come from substituting (21) (for k and $k-1$) and (23) in the equations $J_0V_k = J_1V_{k-1}$, $k = 2,\ldots,n$ and using the results

$$J_0V_k = -2\sum_{i=1}^{N} A_i^{k)}\delta(x-q_i) - 2\sum_{i=1}^{N} B_i^{k)}\delta'(x-q_i) \tag{57}$$

and

$$\begin{aligned} J_1V_{k-1} = &-2\sum_{i,j=1}^{N}\gamma_i A_j^{k-1)}\mathrm{e}^{-|q_i-q_j|}\delta(x-q_i) + \\ &+2\sum_{i,j=1}^{N}\gamma_i B_j^{k-1)}\mathrm{e}^{-|q_i-q_j|}\,\mathrm{sgn}(q_i-q_j)\delta(x-q_i) + \\ &+2\sum_{i,j=1}^{N}\gamma_i A_j^{k-1)}\left(\mathrm{e}^{-|q_i-q_j|} - 1\right)\mathrm{sgn}(q_i-q_j)\delta'(x-q_i) - \\ &-2\sum_{i,j=1}^{N}\gamma_i B_j^{k-1)}\mathrm{e}^{-|q_i-q_j|}\delta'(x-q_i)\,. \end{aligned} \tag{58}$$

Finally the evolutions equations (26) follow from the equation $U_t = J_1 V_n$, and the identities

$$U_t = -2\sum_{i,j=1}^{N}\left(\frac{\partial}{\partial t}\gamma_i\right)\delta(x-q_i) + 2\sum_{i=1}^{N}\gamma_i\left(\frac{\partial}{\partial t}q_i\right)\delta'(x-q_i) \tag{59}$$

and (58) for $k-1=n$.

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