On supersymmetric Q−balls

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We present some facts related to the charged nontopological solutions of nonlinear field equations known as Q–balls. Using simplified field equations from the bosonic sector of the supersymmetric model we discuss an approximate solution with the spherical symmetry.

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1 Introduction

In the last decades there is growing interest in the effects of nonlinearity in the classical and quantum field theories, scattering and coupled states of solitons and soliton–like objects (for example kinks), and the processes of formation and collapse of the domain walls. Such problems are studied in conventional and supersymmetric field theories, in models with the potentials proportional to field in the fourth degree. Topologically nontrivial $U(1)$–states with charge which are called Q–balls were found in this context [1]. A good review of the development in this field is given in Ref.[2]. The Q–balls are described as time–dependent nontopological solutions of equations of motion for complex fields. As such they were defined in the the pioneer paper by Anderson and Derrick [1]. It is shown that stable Q–ball states are not possible for all values of defining parameters. A condition of stable existence may be formulated in the following form. Let in a space of any dimension there is a region where field satisfies the condition $U(\phi^2) - m^2\phi^2 < 0$, where U is a potential, m is a mass of field, then for $\omega^2_{\rm min} \leq \omega^2 \leq m^2$ there are nontopological stable solutions of appropriate equations of motion. The value $\omega_{\rm min}$ is determined by condition imposed on the functions $U(\phi^2)$ and $\omega^2\phi^2$. The problem of stability of the Q–balls under large and small perturbations is still open in a case of large perturbations. For small perturbations the Q–balls are stable, however for large perturbations there is no clear answer to the question of decay of the Q–ball into the plane waves. In the σ–models there are also modifications of the Q–balls known as Q–lumps. Such states are used to study the domain defects, ribbons and domain walls in the supersymmetric models. In the paper [3] a model with two scalar fields was considered. For this model the Witten mechanism allows the breaking of a discrete symmetry. This results in the formation of energetically degenerate domains separated by massive defects. A supersymmetrical model constructed from the chiral superfields was studied in [4]. For this model a domain wall solution which interpolates between supersymmetric vacuum states is obtained.

Using the results of two previous papers there was constructed a model without domain walls but with so called Fermi–balls [5]. In this model the Fermi–balls, which are proposed as candidates for cold matter, arise in the natural way as a result of the softly broken supersymmetry. Further study of such models is done with the temperature and nested defects taken into account. In the paper [6] thermal effects were introduced to study the domain walls hosting domain ribbons. Such a system can be considered as the real bosonic sector of a supersymmetric theory with two different critical temperatures in this model. Each of the two fields has its own critical temperature that drives the symmetry breaking. A generalization of this approach to the supersymmetric model was presented in the paper [7]. The generalized model contains two interacting chiral supersymmetric fields. This allowed to construct string–like domain ribbon defects embedded in a domain wall. The ribbon can support fermion zero modes. By using this results fermions trapped within the ribbon can be described as excitations of these zero modes.

The solutions that describe the domain walls in (3+1)–dimensional supersymmetric theory with distinct discrete vacua are considered in paper the [8]. Two cases were studied: when the domain wall is the BPS–saturated state and when $\frac{1}{2}$ supersymmetry takes a place. The internal structure of the BPS–saturated domain walls was studied in the paper [9]. It is shown that this structure can continuously vary without a change of the wall tension. The shapes of domains in the supersymmetric theory with several scalar superfields are studied in the paper [10]. It was shown that depending on coupling between fields some of domains are unstable and decay into multiple domain walls but others can form intersections in space. Some degenerated domain wall configurations in the generalized Wess–Zumino model with two scalar superfields were investigated in the paper [11]. In this paper the general features inherent in the models with continuously degenerated domain walls were described. In the paper [12] the domain walls in supersymmetric gluodynamics were considered. The phenomenon of the string ending on the domain walls suggested by Witten was described in the framework of the gauge theory. This interpretation was discussed in the supersymmetric theories with the degenerated vacuum states. An intersection of the two domain walls in a supersymmetric context has been considered in the paper [13]. A finite effective length of the intersection region and an energy associated with this process were calculated. A breaking of supersymmetry without the messenger fields was studied in the paper [14]. Under assumption that our world is located on a domain wall and supersymmetry is broken only by other wall which is placed at some distance, the overlap of the wave functions takes the place on our wall. This overlap leads to the mass splitting of physical fields in our world. The domain wall junction in the $\frac{1}{4}$ BPS states was studied in the paper [15]. It is shown that such solutions preserve a single Hermitian supercharge. A study of regularization (renormalization) of (1+1)–supersymmetric solitons in the presence of non-trivial background is given in Ref. [16]. It was shown that for a consistent realization of the regularization (renormalization) it is necessary to find a consistent relation for the cut-offs. The domain walls obtained by embedding the (1+1)–dimensional ϕ^4 kink in higher dimensions were studied in the paper [17]. It was shown that appropriate dimensional regularization allows to avoid intricacies presented in the other regularization schemes. The anomaly in the central charge of supersymmetrical kink that arises from dimensional regularization and reduction was considered in paper [18].

The general mathematical theory of supersymmetric kinks was considered in the papers [19, 20]. For kinks in the Wess–Zumino model with different polynomial superpotentials the explicit forms of vacuum orbits, real algebraic curves, kink curves, energies and conserved supersymmetric charges were presented. The quantum corrections to mass and central charge of kink were calculated and important influence of boundary effects on the mode number cut-off regularization were shown. By using the topologically non-trivial solutions like kinks from bosonic sector of supersymmetric model, the $U(1)$–charged domain wall was described in papers [21, 22]. This charge may be large enough to be important in the problem of the scattering particles by the domain wall. The Q–balls of different types being candidates for the self-interacting dark matter were studied in the paper [23]. It is shown that Q–balls should not evaporate which requires them to have very large charge.

The models with potentials of other type then ϕ^4 are studied in a less degree. The properties of the models with potentials proportional to field in the sixth degree are the most close to the properties of models mentioned above [24]. In most cases such models were investigated using numerical methods. However, for the (1+1)–dimensional problem in the case of the symmetry with the potential

$$U(A,A^*) = m^2|A|^2 - \frac{\lambda^2}{2}\,|A|^4 + \frac{\mu}{3}\,|A|^6\,, \tag{1}$$

an analytic solution has been constructed [25] (see also [2]). Here A is a complex scalar field, m, λ, μ are the real parameters. If, for the model with potential (1) we look for a solution in the form

$$A(x,t) = a(x)\mathrm{e}^{\mathrm{i}\omega t}\,, \tag{2}$$

then we get the equation for $a(x)$ in the following form

$$\partial_x^2 a + \omega^2 a - a\frac{\mathrm{d}U(|a|^2)}{\mathrm{d}|a|^2} = 0\,, \tag{3}$$

where ω is a fixed frequency of field. This equation has a solution

$$a(x) = \frac{m}{\lambda}(1-\varpi^2)^{1/2}a_p(\xi)\,, \tag{4}$$

where $\varpi = \dfrac{\omega}{m}$, $0 \le \varpi \le 1$, $\xi = kx(1-\varpi^2)^{1/2}$. Moreover

$$\begin{gathered} a_p(\xi) = 2\sqrt{3}\big(p\,\mathrm{ch}^2\,\xi - q\,\mathrm{sh}^2\,\xi\big)^{-1/2}, \quad p = 3 + (9-48B)^{1/2}, \quad q = 3 - (9-48B)^{1/2}, \\ B = \frac{\mu m^2}{\lambda}(1-\varpi^2)^{1/2}\,, \quad B < \frac{3}{16}\,. \end{gathered} \tag{5}$$

There exists for this solution a fixed value of frequency $0 < \varpi_{\rm cr} < 1$ for which under the condition $\varpi > \varpi_{\rm cr}$ the soliton–like solution (2) is stable under the small perturbations. In the nonrelativistic limit $\varpi \to 1$ the discrete mode of the spectrum for such solution corresponds to the coupled soliton–like state for nonlinear Schrödinger equation as it was pointed out in [26]. The non-renormalized potential (1) has been used for numerical investigation of a possible construction of a spinning Q–ball solution [27]. Such Q–balls should have a non-zero component of the angular momentum along the z–axis. In Ref. [24] a potential proportional to the field in the sixth degree has been used for the construction of a supersymmetrical model of kinks inside the domain ribbon. In this paper two coupled real scalar fields were considered. For the first of them the potential proportional to the field in the sixth degree was used and for the second: the potential proportional to field in the forth degree. That is why it is interesting to consider a supersymmetric model of charged Q–balls with the potential (1) which contains not real but complex scalar fields in the bosonic sector. We will discuss this problem below.

2 The model

Consider complex chiral superfields

$$\Phi_+ \to {\rm e}^{{\rm i}\alpha}\Phi_+ \,, \quad \Phi_- \to {\rm e}^{-{\rm i}\alpha}\Phi_- \,, \tag{6}$$

where [28]:

$$\Phi_+(y,\,\theta) = A_+ + \sqrt{2}\theta\psi_+ + \theta^2 F_+ \,, \tag{7}$$

$$\Phi_-(y,\,\theta) = A_- + \sqrt{2}\theta\psi_- + \theta^2 F_- \,. \tag{8}$$

Here $y^\mu = x^\mu + {\rm i}\theta\sigma^\mu\theta$. $F_\pm$ are auxiliary complex bosonic fields. The complex scalar bosonic fields are as follows

$$A_\pm = \phi \pm {\rm i}\chi \,, \tag{9}$$

where ϕ and χ are the real functions. The metrics $g_{\mu\nu}$ has a signature $(+---)$. Let us write the action in the form

$$S = \int {\rm d}x\,{\rm d}^2\theta\,{\rm d}^2\bar\theta(\Phi_+\Phi_+^*) + \int {\rm d}x\,{\rm d}^2\theta W(\Phi_+\Phi_-) + \int {\rm d}^2\bar\theta W(\bar\Phi_+\bar\Phi_-) \,, \tag{10}$$

$$S = \int {\rm d}x L \,. \tag{11}$$

Let

$$W(A) = m_1^2\Phi_+\Phi_- + \mu_1^2\Phi_+^2\Phi_-^2 \,, \tag{12}$$

where m_1 and μ_1 are the masses of bosonic fields A_+ and A_- respectively. The Lagrangian of the system described by above action can be written in the form

$$L = L_1 + L_2 + L_3 - V \,, \tag{13}$$

where

$$
\begin{aligned}
L_1 &= -\partial A_+ \partial A_+^* - \partial A_- \partial A_-^* \,, && (14)\\
L_2 &= \tfrac{\mathrm{i}}{2}\left(\partial\psi_+ \sigma \bar\psi_+ - \psi_+ \sigma \partial \bar\psi_+\right) + \tfrac{\mathrm{i}}{2}\left(\partial\psi_- \sigma \bar\psi_- - \psi_- \sigma \partial \bar\psi_-\right), && (15)\\
L_3 &= -\frac{1}{2}\left(\frac{\partial^2 W}{\partial A_+^2}\,\psi_+\psi_+ + \frac{\partial^2 W}{\partial A_-^2}\,\psi_-\psi_- + 2\frac{\partial^2 W}{\partial A_+ \partial A_-}\,\psi_+\psi_-\right), && (16)\\
V &= \left|\frac{\partial W}{\partial A_+}\right|^2 + \left|\frac{\partial W}{\partial A_-}\right|^2 . && (17)
\end{aligned}
$$

Using the equations (7),(8) (9), (12) and the conditions $A_+ = A_-^*$, $\psi_+ = \bar\psi_-$, $\psi_- = \bar\psi_+$ one can write the Lagrangian (13) as

$$
\begin{aligned}
L = {} & -\partial^\mu A_+ \partial_\mu A_+^* - \partial^\mu A_- \partial_\mu A_-^* + \mathrm{i}\partial_\mu(\bar\psi_+ \bar\sigma^\mu \psi_+) + \mathrm{i}\partial_\mu(\bar\psi_- \bar\sigma^\mu \psi_-) + \qquad (18)\\
& + 2\mu_1^2 A_+^2 \bar\psi_+ \bar\psi_+ + 2\mu_1^2 A_+^{*2} \bar\psi_+ \bar\psi_+ - (m_1^2 - 4\mu_1^2 |A_+|^2)(\psi_+ \bar\psi_+ + \bar\psi_+ \psi_+) - \\
& - 2m_1^4 |A_+|^2 + 8 m_1^2 \mu_1^2 |A_+|^4 - 8\mu_1^4 |A_+|^6 \,.
\end{aligned}
$$

3 Bosonic sector

The Q–balls can exist in the bosonic sector of the model. Let us consider the equations of motion restricted to this sector only. In the bosonic sector we make the fermionic fields to vanish. The field equations take the following form

$$
\partial^2 A_+ - m_1^4 A_+ + 8 m_1^2 \mu_1^2 A_+ |A_+|^2 - 12 \mu_1^4 A_+ |A_+|^4 = 0 \,. \qquad (19)
$$

A soliton–like solution for this equation of a form $A_+(x,t) = a(x)\mathrm{e}^{\mathrm{i}\omega t}$ in the case of planar symmetry, for the (1+1)–dimensional problem was mentioned in the first Section of this paper. The equation (19) is more interesting in the case of the spherical symmetry. Here the solution depends on radius r only and equation (19) takes the following form

$$
\frac{\mathrm{d}^2 a}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}a}{\mathrm{d}r} - \big(m^2 - \omega^2\big)a + \lambda^2 a^3 - \frac{3\lambda^4}{8m^2}\,a^5 = 0 \,. \qquad (20)
$$

From the equation (17) one can see an analogy with potential (1) for $m^2 = m_1^4$, $\lambda^2 = 8m_1^2\mu_1^2$ and $\mu = 8\mu_1^2$. In the equation (20) the transition from variables (r, a) to variables (ξ, a_0) can be performed similarly as it was done for the equation (4). This will simplify the equation (20). It can be written as

$$
\frac{\mathrm{d}^2 a_0}{\mathrm{d}\xi^2} + \frac{2}{\xi}\frac{\mathrm{d}a_0}{\mathrm{d}\xi} - a_0 + a_0^3 - B a_0^5 = 0 \,. \qquad (21)
$$

The exact analytic explicit solution for equation (20) or (21) cannot be constructed. Usually this solution is searched with the use of numerical methods. The analytic representations for approximate solutions of this equation were studied in [29]. There were obtained expressions for the $\xi \to 0$ and $\xi \to \infty$ limits. But attempts to construct a particle–like solution by using these approximate expressions were unsuccessful. It was shown that it cannot be done in principle. Therefore here we use another way to construct an approximate solution for equation (21) in the case of spherical symmetry.

Let us consider a thin–wall approximation. In such a case, in the equation (21) a term $\frac{2}{\xi}\frac{\mathrm{d}a_0}{\mathrm{d}\xi}$ may be dropped because of its small value in comparison to the other terms. Denote the solution in this approximation by $h_0(\xi)$. Then the equation for the spherically symmetric Q–ball may be written in the following form

$$\frac{\mathrm{d}^2 h_0}{\mathrm{d}\xi^2} - h_0 + h_0^3 - Bh_0^5 = 0\,. \tag{22}$$

This equation has the similar form as for the case of the planar symmetry. Its solution is known. The only difference here is that a variable ξ changes not along the real axis but from zero to some fixed value that is determined by the approximation of theory. Let us make the following ansatz for the solution of the equation (21)

$$a_0(\xi) = h_0(\xi) + \eta(\xi)\,, \tag{23}$$

where $\eta(\xi)$ is some function. The function h_0 has the same form as the function a_p in the case of planar symmetry in the equation (5). Let us substitute expression (23) into equation (21). Then the nonlinear inhomogeneous equation for the function has the form

$$\begin{aligned}\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} + \frac{2}{\xi}\frac{\mathrm{d}\eta}{\mathrm{d}\xi} - \eta + 3h_0^2\eta + 3h_0\eta^2 + \eta^3 + \frac{2}{\xi}\frac{\mathrm{d}h_0}{\mathrm{d}\xi} - \\ -B\left(5h_0^4\eta + 10h_0^3\eta^2 + 10h_0^2\eta^3 + 5h\eta^4 + \eta^5\right) = 0\,.\end{aligned}$$

This equation is difficult to be solved analytically. For simplicity consider its linear approximation. Additionally, to be able to study the problem analytically, we will drop the terms $3h_0^2\eta$ and $5Bh_0^4\eta$ (note that $B < 1$). Using the explicit form of the function h_0 we get

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} + \frac{2}{\xi}\frac{\mathrm{d}\eta}{\mathrm{d}\xi} - \eta = \frac{4\sqrt{3}(p-q)\,\mathrm{ch}\,\xi\,\mathrm{sh}\,\xi}{\xi(p\,\mathrm{ch}^2\,\xi - q\,\mathrm{sh}^2\,\xi)^{3/2}}\,. \tag{24}$$

The right-hand side of equation (24) grows for $\xi \to \infty$, therefore our analysis is valid for small values of ξ only. This description of expansion of the Q–ball should work at the stage when the radius of Q–ball is not very large and at the same time the thin–wall approximation is valid. It means that the Q–ball has been formed and some time from the beginning of its expansion passed.

4 Approximate solution

To find a general solution η of the inhomogeneous equation (24), let us assume that it is a sum of general solution η_0 for appropriate homogeneous equation and a specific solution η_s of the inhomogeneous equation (24). The solution has the form [30]

$$\eta_0 = C_1\eta_1 + C_2\eta_2 \,, \tag{25}$$

where C_1 and C_2 are constants determined by boundary conditions, $\eta_1 = \exp(-\xi)/\xi$, $\eta_2 = \exp(\xi)/\xi$. A particular solution can be found by the Lagrange method

$$\eta_s = \eta_1 \int \frac{\eta_2 f}{W}\,\mathrm{d}\xi - \eta_2 \int \frac{\eta_1 f}{W}\,\mathrm{d}\xi \,, \tag{26}$$

where f is a right-hand side of equation (24), $W = \dfrac{2}{\xi^2}$ is the Wronskian of functions η_1 and η_2. Finally, η_s may be written in the following form

$$\begin{aligned}\eta_s = & \frac{4\sqrt{3}(p-q)\exp(-\xi)}{2\xi}\int \frac{\exp(\xi)\,\mathrm{ch}\,\xi\,\mathrm{sh}\,\xi}{(p\,\mathrm{ch}^2\,\xi - q\,\mathrm{sh}^2\,\xi)^{3/2}}\,\mathrm{d}\xi - \\ & -\frac{4\sqrt{3}(p-q)\exp(\xi)}{2\xi}\int \frac{\exp(-\xi)\,\mathrm{ch}\,\xi\,\mathrm{sh}\,\xi}{(p\,\mathrm{ch}^2\,\xi - q\,\mathrm{sh}^2\,\xi)^{3/2}}\,\mathrm{d}\xi \,.\end{aligned} \tag{27}$$

The integrals in equation (27) cannot be calculated explicitly. They can be evaluated in the main order of the series of exponential function for small values of ξ. Finally, in the considered approximation, the function η can be written in the following form

$$\eta(\xi) = \frac{C_1\exp(-\xi)}{\xi} + \frac{C_2\exp(\xi)}{\xi} + \frac{4\sqrt{3}\,\mathrm{sh}\,\xi}{\xi(p\,\mathrm{ch}^2\,\xi - q\,\mathrm{sh}^2\,\xi)^{1/2}} \tag{28}$$

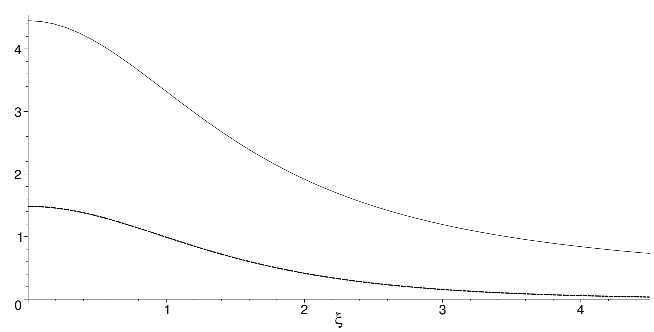

Fig. 1. Dependence of the state amplitude on the variable ξ. Lower curved line — planar symmetry. Upper curved line — spherical symmetry.

For the illustration plots of the functions $a_p(\xi)$ for the planar symmetry and a_0 the spherical symmetry are given in Fig. 1., for $B = \frac{1}{16}$. The shape of the plots very weakly depends on the value of B. The constants C_1 and C_2 we put to zero because of the reasons explained in Ref. [29]. The evaluation of the function $a_0(\xi)$ in the case of the spherical symmetry is very rough. However, the presence of the tendency of the formation Q–ball in comparison to the planar symmetry is noticeable. These solutions, despite different approach, agree with results obtained in Refs. [31, 32].

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