Recoupling theory of many−body quantum theory

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In this paper we sketch the foundations of recoupling theory. Introduction of an indistinguishability principle leads to Pauli Exclusion and confinement. We discuss its application to $SU(3)$ colour.

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1 Introduction

This paper is intended to established the recoupling theory required for the paper [1] given at the sister conference on quantum groups where an extension of graded Lie algebra was presented. Recoupling is an important ingriedent of any many–body quantum theory that deserves to be studied in its own right. Through the indistinguishability principle it gives rise to Pauli's exclusion principle and a mechanism for confinement. For $SU(3)$ this gives a statistical account of quark confinement as first announced in Joyce [2]. This paper is based on the more extensive account given in Joyce [3].

2 Monoidal recouling theory

A many–body quantum theory has the following elements:

1. The symmetry of the physical system is represented by a group G.
2. The fundamental constituent physical systems are represented by finite dimensional irreducible representations.
3. Composition of physical systems is given by tensor product.
4. Recoupling (or statistic) is given by a symmetric monoidal structure.

Items (i) and (ii) assert that the collection of all physical systems exists in the category of representations for the group G which we denote Rep_G. Item (iii) provides a joining operation $\otimes : \mathrm{Rep}_G \times \mathrm{Rep}_G \to \mathrm{Rep}_G$. The last item, (iv), is cyptic so we spend some time developing its meaning.

The recoupling is a collection of natural isomorphisms that reorganize the coupling of any braketed expression of fundamental physical systems into another expression. The elementary recoupling operations (or simply recouplings) are:

– Associativity: $\hat{\alpha} : \otimes(\otimes \times 1) \to \otimes(1 \times \otimes)$ taking $(a, b, c) \mapsto \hat{\alpha}_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$.

– Commutativity: $\hat{\gamma} : \otimes \to \otimes\tau$ where τ is the switch map given by $\tau(a,b) = (b,a)$, taking $(a,b) \mapsto \hat{\gamma}_{a,b} : a \otimes b \to b \otimes a$.

– Left unitarity: $\hat{\lambda} : e \otimes \_ \to \mathrm{id}$ where e is the trivial representation, taking $a \mapsto \hat{\lambda}_a : e \otimes a \to a$.

– Right unitarity: $\hat{\rho} : \_ \otimes e \to \mathrm{id}$ where e is the trivial representation, taking $a \mapsto \hat{\rho}_a : a \otimes e \to a$.

The trivial representation takes on the meaning of a physical system containing no particles. In other words the vacuous physical system.

Composite physical systems are graded as follows. Two physical systems a and b are equivalent ($a \sim b$) if there exists a composite system w such that $a, b \subset w$. Let $[a] = \{b : b \text{ is a composite system and } a \sim b\}$. The collection of these equivalence classes $\mathbb{A}$ is an Abelian group with $[a] + [b] = [a \otimes b]$, $0 = [e]$ and $-[a] = [a^*]$. For example, if $G = SU(n)$ then $\mathbb{A} = \mathbb{Z}_n$, the cyclic group on n letters. For spin this is precisely the Bose/Fermi spin grade.

The outcomes of observations (represented by intertwing operators) must be preserved. Thus the recouplings must be natural transformations in the category theoretic sense. In Joyce [3] it is shown that the most general form is given by

$$\hat{\alpha}_{a,b,c}(a \otimes b) \otimes c = \alpha_{m,n,p} a \otimes (b \otimes c)\,, \tag{1}$$

$$\hat{\gamma}_{a,b} a \otimes b = \gamma_{m,n} b \otimes a\,, \tag{2}$$

$$\hat{\lambda}_a e \otimes a = \lambda_m a\,, \tag{3}$$

$$\hat{\rho}_a a \otimes e = \rho_m a\,, \tag{4}$$

where a, b, c are physical systems of grade m, n, p. Thus a recopupling is represented by four maps $\alpha : \mathbb{A}^3 \to S^1$, $\gamma : \mathbb{A}^2 \to S^1$ and $\lambda, \rho : \mathbb{A} \to S^1$.

In the recoupling process there are often many paths which recouple two physical expressions. For example, figure 1 gives two alternative paths from $((a \otimes b) \otimes c) \otimes d$

Fig. 1. The pentagon condition

to $a \otimes (b \otimes (c \otimes d))$. On says the recoupling is coherence if any two such paths give the same recoupling. The Mac Lane coherence theorem asserts that this holds if and only if the following *symmetric monoidal constraint* conditions hold.

$$\alpha_{m+n,p,q}\alpha_{m,n,p+q} = \alpha_{m,n,p}\alpha_{m,n+p,q}\alpha_{n,p,q} \tag{5}$$

$$\alpha_{m,n,p}\gamma_{m,n+p}\alpha_{n,p,m} = \gamma_{m,n}\alpha_{n,m,p}\gamma_{m,p} \, , \tag{6}$$

$$\gamma_{m,n}\gamma_{n,m} = 1 \, , \tag{7}$$

$$\rho_m = \alpha_{m,0,n}\lambda_n \, , \tag{8}$$

for all $m, n, p, q \in \mathbb{A}$. When α, λ and ρ are trivial then the gamma's are the commutation factors found in graded Lie algebra [4, 5, 6]. An immediate consequent of this is $\gamma_{0,0} = 1$. Some important examples: If $\mathbb{A} = \mathbb{Z}_1$ then all recoupling phases are unit with the possible exception $\lambda_0 = \rho_0$. For $SU(2)$ spin $\mathbb{A} = \mathbb{Z}_2$ and one requires $\gamma_{1,1} = -1$. In this case one can take all other recoupling phases to be unit. For $SU(3)$ colour, the gauge symmetry underlying QCD, the composite states are graded using triality as given by $\mathbb{A} = \mathbb{Z}_3$. In particular, the quarks are of grade one with $\gamma_{1,1} = -1$, and the anti–quarks are of grade two with $\gamma_{2,2} = -1$. No such solution to the symmetric monoidal constraints exist satisfying these two requirements. Thus the recoupling path determines different phases factors for colour as we shall see in the next chapter

3 Indistinguishability and statistics

The coupling process provides an order in which to couple particles together to form composite systems. In the monoidal situation it is not clear, for example, in the expression $(a \otimes b) \otimes (c \otimes d)$ whether $a \otimes b$ is formed before or after $c \otimes d$. To resolve this ambiguity one needs to break the pentagon condition of figure 1. Thus the recoupling is weakened to a symmetric premonodal structure as in Joyce [7]. The order of coupling is best visualized with coupling trees. These are planar rooted binary trees with a linear ordering of the internal nodes such that each loop free path from the root to any leaf is increasing. The pentagon diagram of figure 1 becomes the hexagon given in figure 2.

A measure of the failure of the pentagon is given by the deformativity recoupling $\xi : \mathbb{A}^4 \to S^1$ defined by

$$\xi_{m,n,p,q} = \frac{\alpha_{m,n,p}\alpha_{m,n+p,q}\alpha_{n,p,q}}{\alpha_{m+n,p,q}\alpha_{m,n,p+q}} \, . \tag{9}$$

Thus the pentagon condition is replaced by the weaker conditions:

$$\xi_{m,n,p,q}\xi_{p,q,m,n} = 1 \tag{10}$$

$$\lambda_m = \alpha_{0,m,n}\lambda_{m+n} \tag{11}$$

$$\rho_{m+n} = \alpha_{m,n,0}\rho_n \, . \tag{12}$$

Again one can keep track of the different paths by using the preorder category of coupling trees as described in Joyce [7].

Fig. 2. The deformed pentagon condition represented by coupling trees.

For a symmetric premonoidal recoupling $\xi_{m,n,m,n} = \gamma_{m+n,m+n}\gamma_{m,m}\gamma_{n,n}$ showing that the pentagon must fail for $SU(3)$ colour Bose–Fermi recouplings. Since the pentagon is no longer required to hold we can give a Bose–Fermi recoupling for $SU(3)$ colour. In fact a Bose–Fermi recoupling satisfying $\gamma_{m,m} = -1$ whenever $m \neq 0$ is given by the following choice.

$$\alpha_{m,n,p} = \begin{cases} 1 & : m = 0, n = 0, p = 0 \text{ or } m + n = 0\,, \\ -1 & : \text{otherwise}\,. \end{cases} \tag{13}$$

$$\gamma_{m,n} = \begin{cases} 1 & : m = 0 \text{ or } n = 0\,, \\ -1 & : \text{otherwise}\,. \end{cases} \tag{14}$$

and $\lambda_m = \rho_m = 1$ for all $m, n, p, q \in \mathbb{A}$. A simple calculation shows that deformativity is given by

$$\xi_{m,n,p,q} = \begin{cases} 1 & : m = 0, n = 0, p = 0, q = 0, m + n = 0 \text{ or } p + q = 0\,, \\ -1 & : \text{otherwise}\,. \end{cases} \tag{15}$$

A fundamental principle in many–body quantum theory is that of indistinguishability. In our context the principle states that given a recoupling r between two identical composite systems w then any state $\psi \in w$ of the system must satisfy $r\psi = \psi$. For example, for $w = a \otimes a$ where a is of grade m we have that $\tau\psi = \gamma_{m,m}\psi$. Hence if $\gamma_{m,m} = 1$ then ψ is symmetric, otherwise $\gamma_{m,m} = -1$ and ψ is anti–symmetric. For a Bose–Fermi recoupling this gives Pauli's exclusion principle.

For a $SU(3)$ Bose–Fermi recoupling as required for colour dynamics with $w = (a \otimes b) \otimes (c \otimes d)$ where a is of grade m, etc, we obtain $(1 - \xi_{m,n,p,q})\psi = 0$. Thus the composite can never exist whenever $\xi_{m,n,p,q} \neq 1$. For four free quarks ($\xi_{1,1,1,1} = -1$)

this is precisely the case. Thus we have an alternative statistical explanation of quark confinement. This is a weaker version of a proposal by Günaydin and Gürsey [8] where the associativity was required to vanish.

Finally one should note that this mechanism can be avoid by weakening the notion of composition of physical systems. Usually it is assumed to be tensor product which forces confinement mechanisms upon us. The paper [1] presented at the sister conference on quantum groups discuss the notion of ω–monoidalness [9] as an alternative to the premonoidal structure.

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