Lorentz−like formulation of Galilean field theories

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We construct non-relativistic Lagrangian field models by enforcing Galilean covariance with a $(4,1)$ Minkowski manifold followed by a projection onto the $(3,1)$ Newtonian space–time. We discuss scalar, Fermi and gauge fields, as well as interactions between some of these fields. The Galilean covariant formalism provides an elegant construction of the Lagrangians which describe the electric and magnetic limits of Galilean electromagnetism. As further examples of scalar fields, we discuss various models of fluids and superfluids. Then, we turn to linear wave equations, and consider the Dirac Lagrangian which allows one to retrieve the Lévy-Leblond wave equations. We examine the situation where the Fermi field interacts with an abelian gauge field. Finally, we study the Bhabha equations for spins 0 and 1.

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1 Introduction

Although Galilean relativity has been superseded by Einstein's theory, there exists a wealth of low–energy systems, particularly in condensed matter physics and low–energy nuclear physics, where any new method or result concerning Galilean invariance is likely to be useful. Landau's theory of superfluid state of ^{4}He is but an example. The general program presented hereafter consists in investigating physical applications of a metric formulation of Galilei invariance such that one can use tensor analysis, as it is done in relativistic physics. Hereafter, we summarize Refs. [1] − [6], which follow the approach in Ref. [7]. Similar procedures can be found in [8].

We define a five–dimensional *Galilei-vector* to be such that a boost acts on it as

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{V}t\,, \\ t' &= t\,, \\ s' &= s - \mathbf{V}\cdot\mathbf{x} + \tfrac{1}{2}\mathbf{V}^2 t\,, \end{aligned} \tag{1}$$

with relative velocity $\mathbf{V}$. The scalar product

$$(A|B) = A^\mu B_\mu \equiv \mathbf{A}\cdot\mathbf{B} - A_4 B_5 - A_5 B_4\,, \tag{2}$$

of two Galilei–vectors A and B is invariant under transformation (1). Therefore, we define the Galilean metric as

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} . \tag{3}$$

Clearly this may be diagonalized to $\mathrm{diag}(1,1,1,-1,1)$, so that our starting point is in fact the fifteen–dimensional inhomogeneous Lorentz group in $(4,1)$ space–time. The fact the algorithm allows one to retrieve Galilei–invariant equations amounts to the fact that this Lorentz group contains the eleven–dimensional central extension of the Galilei group.

We can write Eq. (1) as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \,, \tag{4}$$

where μ denotes the row and ν the column (so that $\Lambda^{\mu}{}_{\nu}$ is the $(\mu\nu)$–entry), or

$$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \\ x'^4 \\ x'^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -V_1 & 0 \\ 0 & 1 & 0 & -V_2 & 0 \\ 0 & 0 & 1 & -V_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -V_1 & -V_2 & -V_3 & \frac{1}{2}\mathbf{V}^2 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} . \tag{5}$$

Galilean one–forms transform as

$$x'_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu} \,, \tag{6}$$

where μ now denotes the column and ν the row (that is $\Lambda_{\mu}{}^{\nu}$ is the $(\nu\mu)$–entry), or

$$(x'_1, x'_2, x'_3, x'_4, x'_5) = (x_1, x_2, x_3, x_4, x_5) \begin{pmatrix} 1 & 0 & 0 & V_1 & 0 \\ 0 & 1 & 0 & V_2 & 0 \\ 0 & 0 & 1 & V_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ V_1 & V_2 & V_3 & \frac{1}{2}\mathbf{V}^2 & 1 \end{pmatrix} , \tag{7}$$

the matrix elements are calculated from $x'_{\mu} = g_{\mu\alpha} x'^{\alpha} = \overbrace{g_{\mu\alpha} \Lambda^{\alpha}{}_{\beta} g^{\beta\nu}}^{\Lambda_{\mu}{}^{\nu}} \, x_{\nu}$. Note that the units of s are $L^2 T^{-1}$.

Throughout these notes we utilize the Galilei–vectors $(x^1, \ldots, x^5)$ with each component having units of length:

$$(x^1, \ldots, x^5) = \left(\mathbf{x}, v_4 t, \frac{s}{v_5} \right) . \tag{8}$$

For a real field $\tilde{\phi}$, the projection is defined as

$$\tilde{\phi}(x) \equiv \phi(\mathbf{x}, t) + a_0 s \,. \tag{9}$$

For a complex field $\tilde{\psi}$ we use the definition:

$$\tilde{\psi}(x) \equiv \mathrm{e}^{\mathrm{i}a_0 s}\psi(\mathbf{x}, t)\,. \tag{10}$$

Hereafter, we take $v_4 = v_5$ in Eq. (8), except in Section 3, where $v_4 = c$ and $v_5 = 1$. In Sections 2, 6 and 7, we use Eq. (10) with $a_0 = -m$, but we use $a_0 = -mc$ in Section 3. In Sections 4 and 5, where we consider real scalar fields, we utilize Eq. (9) with $a_0 = -1$.

From Eq. (8), and the following definition for the five–momentum

$$p_\mu \equiv -\mathrm{i}\partial_\mu = \left(-\mathrm{i}\nabla, -\mathrm{i}\frac{\partial_t}{v_4}, -\mathrm{i}v_5\partial_s\right), \tag{11}$$

together with $E = \mathrm{i}\partial_t$, as well as $m = \mathrm{i}\partial_s$, we obtain

$$\begin{aligned} p_\mu &= \left(\mathbf{p}, -\frac{E}{v_4}, -mv_5\right), \\ p^\mu &= g^{\mu\nu}p_\nu = \left(\mathbf{p}, mv_5, \frac{E}{v_4}\right). \end{aligned} \tag{12}$$

Thereupon the mass does not enter as an external parameter, but as a remnant of the fifth component of the particle's momentum, starting from an apparently massless theory in five dimensions!

In Section 2, we consider the Klein–Gordon Lagrangian in five dimensions, and see that the reduction leads to the Schrödinger field. In Section 3, we retrieve the equations of 'Galilean electromagnetism', first described in 1973 by Le Bellac and Lévy-Leblond, and we determine the Lagrangians which provide those equations. In Sections 4 and 5, we describe the Euler equations for fluids, and present some models for superfluids, respectively. The Dirac equation turns out to reduce to the Lévy-Leblond equations, as shown in Section 6, and linear Bhabha wave equations for particles with spins 0 and 1 are presented in Section 7.

2 Klein−Gordon Lagrangian and Schrödinger field

In this section, let us consider a simple example of a relativistically invariant wave equation, obtained from the Galilean Klein–Gordon Lagrangian:

$$\mathcal{L}_{\mathrm{GKG}} = -\frac{1}{2m}\left(\partial^\mu\Phi^*\partial_\mu\Phi - k^2|\Phi|^2\right) - V\left(|\Phi|\right). \tag{13}$$

By 'Galilean', we mean that the field is defined on the extended space–time, and this model will describe non-relativistic physics, once we have defined the embedding defined in Eq. (8). The Euler–Lagrange field equation taken with respect to Φ^* gives the scalar equation

$$\frac{1}{2m}(\partial^\mu\partial_\mu + k^2)\Phi = \frac{\delta V}{\delta\Phi^*}\,. \tag{14}$$

With the embedding defined in Eqs. (8) and (10), if we absorb k into the energy operator, then this becomes the Schrödinger equation:

$$\mathrm{i}\partial_t\varphi = -\frac{1}{2m}\nabla^2\varphi + \frac{\delta V}{\delta\varphi^*}\,. \tag{15}$$

(The more familiar Schrödinger equation, $\mathrm{i}\partial_t\varphi = -\frac{1}{2m}\nabla^2\varphi + \mathcal{V}(r)\varphi$, may be obtained by restricting the potential to $V(|\varphi|) = -\mathcal{V}(r)|\varphi|^2$.) Now, if, rather than first finding the Euler–Lagrange equations, we begin by substituting Eqs. (8) and (10) into the Lagrangian $\mathcal{L}_{\mathrm{GKG}}$, then it becomes

$$\mathcal{L} = -\frac{1}{2m}\,|\nabla\varphi|^2 - \frac{\mathrm{i}}{2}\,\big((\partial_t\varphi^*)\varphi - \varphi^*\partial_t\varphi\big) - V\big(|\varphi|\big)\,. \tag{16}$$

The variation with respect to φ^* leads, once again, to the Schrödinger equation (15). In the next sections, we will see that for other models, the resulting equations of motion are not the same, depending on the order in which we perform the reduction or compute the Euler–Lagrange equations. For the quartic self-interaction, $V(|\Phi|) = \frac{1}{2}\lambda\Phi^4$, Eq. (15) becomes

$$\mathrm{i}\partial_t\varphi = -\frac{1}{2m}\nabla^2\varphi + \lambda|\varphi|^2\varphi\,. \tag{17}$$

This is referred to as the non-linear Schrödinger equation or, to condensed matter physicists, as the Gross–Pitaevskii equation.

3 Galilean electromagnetism

In this section, we turn to the 'Galilean electromagnetism' described thirty years ago by Le Bellac and Lévy-Leblond [9]. Hereafter, we retrieve its Galilei–invariant 'electric' and 'magnetic' limits by using the tensorial form of Maxwell equations, and determine the Lagrangian densities from which the following field equations are derived. The Galilean Maxwell equations read:

$$\begin{aligned}
&\nabla\cdot\mathbf{B} = 0\,,\\
&\nabla\cdot\mathbf{E}_m = \frac{1}{\epsilon_0}\,\rho_m\,,\\
&\nabla\times\mathbf{B} = \mu_0\mathbf{J}\,,\\
&\nabla\times\mathbf{E}_m = -\partial_t\mathbf{B}
\end{aligned} \tag{18}$$

for the 'magnetic' limit, and

$$\begin{aligned}
&\nabla\cdot\mathbf{B} = 0\,,\\
&\nabla\cdot\mathbf{E}_e = \frac{1}{\epsilon_0}\,\rho_e\,,\\
&\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial_t\mathbf{E}_e\,,\\
&\nabla\times\mathbf{E}_e = \mathbf{0}
\end{aligned} \tag{19}$$

for the 'electric' limit. Note that the displacement current term is missing in the third line of Eq. (18), and that the Faraday induction term does not appear in the last line of Eq. (19). The purpose of Le Bellac and Lévy-Leblond was to write down the laws of electromagnetism by enforcing Galilean relativity rather than Einstein's relativity. Therefore, the equations above could have been formulated during the pre-relativity era.

The authors of Ref. [9] have observed that the Lorentz transformation of a four–vector $(u^0, \mathbf{u})$:

$$\begin{aligned} u'^0 &= \gamma\left(u^0 - \frac{1}{c}\,\mathbf{V}\cdot\mathbf{u}\right), \\ \mathbf{u}' &= \mathbf{u} - \gamma\frac{\mathbf{V}}{c}\,u^0 + \frac{\mathbf{V}}{\mathbf{V}^2}\,(\gamma-1)\mathbf{V}\cdot\mathbf{u}\,, \end{aligned} \tag{20}$$

where $\gamma \equiv \big(1-\mathbf{V}^2/c^2\big)^{-1/2}$, with relative velocity $\mathbf{V}$ and speed of light in the vacuum c, admits two well-defined Galilean limits. One limit is related to largely time–like vectors, with $u'^0 = u^0$ and $\mathbf{u}' = \mathbf{u} - \big(\mathbf{V}/c\big)u^0$, and it corresponds to the electric limit. The second limit is for largely space–like vectors, which satisfy $u'^0 = u^0 - c^{-1}\mathbf{V}\cdot\mathbf{u}$ and $\mathbf{u}' = \mathbf{u}$, and is associated with the magnetic limit. The magnetic limit corresponds to systems were the magnetic field is much greater than the electric field. The opposite situation, where the electric field is large, corresponds to the electric limit. In addition to the field equations, Le Bellac and Lévy-Leblond have determined various field transformations, but they have not discussed which Lagrangians provide the two Galilean limits [9]. We will show hereafter that the two Lagrangians have the same form, and both involve different auxiliary fields, which are set equal to zero once the equations of motion have been obtained.

In this section, we utilize Eq. (8) in such a way that all the components have units of length:

$$(\mathbf{x}, t) \hookrightarrow x^\mu = (x^1, \cdots, x^5) \equiv (\mathbf{x}, ct, s)\ , \tag{21}$$

where c has the dimension of a velocity. Eq. (21) implies that we must replace Eq. (12) with

$$\partial_\mu = \left(\nabla, \frac{1}{c}\partial_t, \partial_s\right), \qquad -\mathrm{i}\partial_\mu = \left(\mathbf{p}, -\frac{E}{c}, -mc\right), \tag{22}$$

so that $p^4 = -p_5 = mc$ and $p^5 = -p_4 = E/c$. Thus, we obtain $\partial_s = -\mathrm{i}mc$.

The five–dimensional Galilean Lagrangian of electromagnetism, that is, the Maxwell Lagrangian interacting with an external five–current J_μ, is given by

$$\mathcal{L}_{\mathrm{GEM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{\epsilon_0 c}J_\mu A^\mu\,, \tag{23}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{24}$$

If we calculate the Euler–Lagrange equations for the gauge fields A^μ, we obtain the field equations

$$\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0\,, \tag{25}$$

and

$$\partial_\mu F^{\mu\nu} = -\frac{1}{\epsilon_0 c} J^\nu. \tag{26}$$

For later convenience, let us introduce the parameter μ_0:

$$\mu_0 \epsilon_0 = \frac{1}{c^2}\,. \tag{27}$$

In Ref. [2], we have shown that these equations lead to the electric and magnetic limits, as identified by Le Bellac and Lévy-Leblond in Ref. [9]. Moreover, we have noticed that the transformation laws of A_μ, J_μ and the electromagnetic field can be retrieved very naturally with our five–dimensional algorithm. However, one cannot determine the Lagrangians which provide the two Galilean limits, Eqs. (18) and (19), by simply defining the fields as in Ref. [2]. Indeed, if we substitute Eqs. (39) and (40) of Ref. [2] into Eq. (23), then we find the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\mathbf{B}_e^2 + \mu_0\epsilon_0 \partial_t \mathbf{A}_e \cdot \nabla\phi_e + \tfrac{1}{2}\mu_0^2\epsilon_0^2 (\partial_t \phi_e)^2 - \mu_0 \mathbf{J}_e \cdot \mathbf{A}\,, \tag{28}$$

which clearly does not lead to Eq. (19). For the magnetic limit, the situation is even worse, because the electric field does not appear at all within the Lagrangian. Substituting equations (49) and (50) of Ref. [2] into the Lagrangian given by Eq. (23), we find

$$\mathcal{L} = -\tfrac{1}{2}\mathbf{B}_m^2 - \mu_0 \mathbf{J}_m \cdot \mathbf{A}\,. \tag{29}$$

It appears that in order to construct two such Lagrangians, with one leading to the electric limit, and the other leading to the magnetic limit, one needs to introduce auxiliary fields. The latter are used throughout the computation of the Euler–Lagrange equations, and only then they may be eliminated. Hereafter, we utilize the formalism based on a Minkowski space–time in $(4,1)$ dimensions to find that the ensuing field equations may be obtained from a single Lagrangian in $(4,1)$ dimensions which reduces to two different Lagrangians in $(3,1)$ space–time. It turns out that one set of auxiliary fields leads to the electric limit, whereas a complementary set of auxiliary fields provides the magnetic limit. Whereas the form of this Lagrangian, as well as the physical and auxiliary fields, are suggested very naturally by using the Minkowski space in $(4,1)$, it would be far from obvious without this formalism. This is done by defining the five–potential as

$$A_\mu(x) = \big(\mathbf{A}(\mathbf{x},t), -\phi_m(\mathbf{x},t), -\phi_e(\mathbf{x},t)\big)\,. \tag{30}$$

That these fields do not depend on s can be traced back, using Eq. (12), to the fact that they describe a massless field, so that $i\partial_s = m = 0$. The five–current J_μ is defined similarly,

$$J_\mu = \big((\mathbf{J}(\mathbf{x},t), -c\rho_m(\mathbf{x},t), -c\rho_e(\mathbf{x},t)\big)\,, \tag{31}$$

where each component is independent of s.

Let us denote the components of the field strength tensor by

$$F_{\mu\nu} = \begin{pmatrix} 0 & cB_3 & -cB_2 & E_{m1} & E_{e1} \\ -cB_3 & 0 & cB_1 & E_{m2} & E_{e2} \\ cB_2 & -cB_1 & 0 & E_{m3} & E_{e3} \\ -E_{m1} & -E_{m2} & -E_{m3} & 0 & a \\ -E_{e1} & -E_{e2} & -E_{e3} & -a & 0 \end{pmatrix}, \tag{32}$$

so that, from Eqs. (24) and (30), we find

$$\begin{aligned} a &= -\frac{1}{c}\,\partial_t\phi_e\,, \\ c\mathbf{B} &= \nabla\times\mathbf{A}\,, \\ \mathbf{E}_m &= -\nabla\phi_m - \frac{1}{c}\,\partial_t\mathbf{A}\,, \\ \mathbf{E}_e &= -\nabla\phi_e\,. \end{aligned} \tag{33}$$

Then, by substituting this into Eq. (23), the Galilean version of Lagrangian $\mathcal{L}_{\mathrm{EM}}$ reads

$$\mathcal{L}_{\mathrm{GEM}} = -\frac{1}{2}\,c^2\mathbf{B}^2 + \mathbf{E}_m\cdot\mathbf{E}_e + \frac{1}{2c^2}\,(\partial_t\phi_e)^2 + \frac{1}{\epsilon_0 c}\mathbf{J}\cdot\mathbf{A} - \frac{1}{\epsilon_0}\rho_m\phi_e - \frac{1}{\epsilon_0}\rho_e\phi_m\,. \tag{34}$$

This is the central result of this section.

Once again, let us recall that there are *not* two kinds of physical electric fields, $\mathbf{E}_e$ and $\mathbf{E}_m$. Only one is taken to be the physical field, while the other is an auxiliary field, in the respective (i.e. electric or magnetic) limit. If we compute the Euler–Lagrange equations with respect to the fields ϕ_e, ϕ_m and $\mathbf{A}$, we find

$$\nabla\cdot\left(-\nabla\phi_m - \frac{1}{c}\,\partial_t\mathbf{A}\right) = \frac{1}{\epsilon_0}\rho_m + \frac{1}{c^2}\,\partial_{tt}\phi_e\,, \tag{35}$$

$$\nabla\cdot(-\nabla\phi_e) = \frac{1}{\epsilon_0}\rho_e\,, \tag{36}$$

and

$$c\nabla\times\mathbf{B} = \frac{1}{\epsilon_0 c}\,\mathbf{J} + \frac{1}{c}\,\partial_t(-\nabla\phi_e), \tag{37}$$

respectively.

In order to retrieve Le Bellac and Lévy-Leblond's magnetic limit [9], we define the auxiliary quantities ϕ_e and ρ_e as

$$\phi_e = 0\,, \quad \rho_e = 0\,, \tag{38}$$

with $\mathbf{E}_m$ given by Eq. (33), so that Eqs. (35) and (37) reduce to the Gauss's law,

$$\nabla\cdot\mathbf{E}_m = \frac{1}{\epsilon_0}\rho_m\,, \tag{39}$$

as well as

$$\nabla\times\mathbf{B} = \mu_0 \mathbf{J}\,, \tag{40}$$

respectively. The Eq. (36) vanishes identically.

The electric limit is obtained by defining

$$\phi_m = 0\,, \quad \rho_m = 0\,, \tag{41}$$

with $\mathbf{E}_e$ given by Eq. (33). From Eqs. (36) and (37) we find Gauss's law,

$$\nabla\cdot\mathbf{E}_e = \frac{1}{\epsilon_0}\rho_e\,, \tag{42}$$

and

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial_t\mathbf{E}, \tag{43}$$

respectively. From Eq. (35), we obtain the Lorentz gauge condition:

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\partial_t\phi_e = 0\,. \tag{44}$$

Now let us turn to the homogeneous Eqs. (25), that is,

$$\begin{aligned}
&\nabla\cdot\mathbf{B} = 0\,,\\
&\nabla\times\mathbf{E}_m + c\partial_4\mathbf{B} = \mathbf{0}\,,\\
&\nabla\times\mathbf{E}_e + c\partial_5\mathbf{B} = \mathbf{0}\,,\\
&\nabla a - \partial_4\mathbf{E}_e + \partial_5\mathbf{E}_m = \mathbf{0}\,.
\end{aligned} \tag{45}$$

The first of these clearly leads to

$$\nabla\cdot\mathbf{B} = 0\,, \tag{46}$$

in both limits. The second leads, in the magnetic limit defined by Eqs. (33) and (38), to

$$\nabla\times\mathbf{E}_m = -\partial_t\mathbf{B}\,, \tag{47}$$

and vanishes identically in the electric limit, defined by Eq. (41). The third equation gives

$$\nabla\times\mathbf{E}_e = \mathbf{0}\,, \tag{48}$$

in the electric limit, Eq. (41), and vanishes identically in the magnetic limit, Eq. (38). The fourth equation leads to an identically vanishing result. To summarize, the magnetic limit of the Maxwell Eqs. (18) is retrieved from Eqs. (39), (40), (46) and (47). The electric limit equations (19) are obtained by collecting Eqs. (42), (43), (46) and (48).

4 Fluids: Euler equation

In the next two sections, we summarize some results of Refs. [3, 4]. We consider real scalar fields, and consider them in the context of fluids and superfluids models. In the present section, we show the covariant five–dimensional form of the Euler equation of fluids.

Let us start with the functional Lagrangian :

$$\tilde{\mathcal{L}}[\tilde\rho,\tilde\phi] = -\tfrac{1}{2}\tilde\rho\partial_\mu\tilde\phi\partial^\mu\tilde\phi - V(\tilde\rho)\,. \tag{49}$$

Euler–Lagrange equation for $\tilde\rho$ is $\frac{1}{2}\partial_\mu\tilde\phi\partial^\mu\tilde\phi + V'(\tilde\rho) = 0$. Define the embedding in Eqs. (8) and (9), with $v_4 = v_5$ and $a_0 = -1$, as well as

$$\tilde\rho(x) \equiv \rho(\mathbf{x},t)\,. \tag{50}$$

This gives

$$\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi = -V'. \tag{51}$$

If we compute the gradient of this expression, we find

$$(\nabla\phi\cdot\nabla)\nabla\phi + \partial_t(\nabla\phi) = -\nabla(V')\,. \tag{52}$$

With $\mathbf{v} = \nabla\phi$ and $\nabla(V') = \rho^{-1}\nabla p$, where p denotes the pressure, we find the Euler equation,

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p\,. \tag{53}$$

When we compute the Euler–Lagrange with respect to $\tilde\phi$, we find the continuity equation:

$$\partial_t\rho + \nabla(\rho\nabla\phi) = 0\,. \tag{54}$$

Now consider the following Lagrangian, which depends on a complex field:

$$\tilde{\mathcal{L}}[\tilde\psi,\tilde\psi^*] = k_1\big(\partial_\mu\tilde\psi\partial^\mu\tilde\psi^* - V(|\tilde\psi|)\big)\,. \tag{55}$$

If we define the real fields $\tilde\rho$ and $\tilde\phi$ by using the Madelung prescription,

$$\tilde\psi \equiv \sqrt{\tilde\rho}\,\mathrm{e}^{\mathrm{i}\tilde\phi}, \tag{56}$$

then the Lagrangian becomes

$$\tilde{\mathcal{L}}[\tilde\rho,\tilde\phi] = k_1\left(\tilde\rho\partial_\mu\tilde\phi\partial^\mu\tilde\phi + \frac{1}{4\tilde\rho}\partial_\mu\tilde\rho\partial^\mu\tilde\rho - V\left(\sqrt{\tilde\rho}\right)\right), \tag{57}$$

or

$$\tilde{\mathcal{L}} = k_1\left(\tilde\rho\partial_\mu\tilde\phi\partial^\mu\tilde\phi - \overline{V}(\tilde\rho)\right), \tag{58}$$

where $\overline{V} \equiv V - \dfrac{1}{4\tilde\rho}\,\partial_\mu\tilde\rho\,\partial^\mu\tilde\rho$. Clearly, Eq. (58) coincides with Eq. (49).

5 Models for superfluidity

5.1 Barotropic irrotational fluid

In this section, we discuss some models proposed by Takahashi to describre superfluidity [7]. The model for compressible irrotational barotropic fluids with pressure proportional to the square of the mass density based on

$$\tilde{\mathcal{L}} = \frac{\rho_0}{8v_0^2}\left(\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - 2v_0^2\right)^2 . \tag{59}$$

The $\tilde{\phi}$ is related to velocity potential. The equation of motion with respect to $\tilde{\phi}$ is

$$\partial_\mu\partial^\mu\tilde{\phi} - \frac{1}{2v_0^2}(\partial_\mu\partial^\mu\tilde{\phi})(\partial_\nu\tilde{\phi}\partial^\nu\tilde{\phi}) - \frac{1}{v_0^2}(\partial^\mu\tilde{\phi})(\partial^\nu\tilde{\phi})(\partial_\mu\partial_\nu\tilde{\phi}) = 0\,. \tag{60}$$

If we take $v_4 = v_5$ in Eq. (8), and Eq. (9) with $a_0 = -1$, then Eq. (59) reduces to

$$\mathcal{L} = \frac{\rho_0}{2v_0^2}\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi - v_0^2\right)^2 . \tag{61}$$

With similar definitions, the equation of motion, Eq. (60), becomes

$$v_0^2\nabla^2\phi - \partial_t^2\phi = \nabla^2\phi\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi\right) + \tfrac{1}{2}\nabla\phi\cdot\nabla\left(\nabla\phi\cdot\nabla\phi\right) + 2\nabla(\partial_t\phi)\cdot\nabla\phi\,, \tag{62}$$

which is the equation (5.40) obtained by Takahashi [7]. Some solutions for one–dimensional case have been found in [4]. A symmetry analysis leads to vector fields

$$\begin{aligned}
\mathbf{v}_1 &= \partial_t\,,\\
\mathbf{v}_2 &= \partial_\phi\,,\\
\mathbf{v}_3 &= \partial_x\,,\\
\mathbf{v}_4 &= \left(t - \frac{\phi}{v_0^2}\right)\partial_\phi - \frac{x}{2v_0^2}\partial_x\,,\\
\mathbf{v}_5 &= x\partial_\phi + t\partial_x\,,\\
\mathbf{v}_6 &= x\partial_x + t\partial_t + \phi\partial_\phi\,.
\end{aligned} \tag{63}$$

The vectors $\mathbf{v}_1 + c\mathbf{v}_3$ and $\mathbf{v}_3 + c\mathbf{v}_2$ generate travelling–wave solutions. The subgroup generated by $\mathbf{v}_6$ lead to scale–invariant solutions $\phi(x,t) = \dfrac{1}{3}\dfrac{x^2}{t} + (v_0^2+k^2)t \pm \mathrm{i}\sqrt{\dfrac{2}{3}}\,kx$. The $\mathbf{v}_5$–invariant solutions have the form $\phi(x,t) = \dfrac{x^2}{2t} - \dfrac{k_1}{2}\ln t + v_0^2 t - \dfrac{k_2}{2}$ and from $\mathbf{v}_4$ we find $\phi(x,t) = \dfrac{cx^2}{t+k} + v_0^2 t$ (where $c = 1/2$ or $1/3$).

Now consider the Lagrangian

$$\tilde{\mathcal{L}} \propto D_\mu\tilde{\psi}D^\mu\tilde{\psi}^* \equiv (\partial_\mu\tilde{\psi} + \mathrm{i}\tilde{A}_\mu\tilde{\psi})(\partial^\mu\tilde{\psi}^* - \mathrm{i}\tilde{A}^\mu\tilde{\psi}^*)\,, \tag{64}$$

with $\tilde{A}_\mu = k\partial_\mu\tilde{\rho}$. The Euler–Lagrange equation for $\tilde{\psi}^*$ with Eqs. (8) and (10), and $\tilde{\rho}(x \equiv \rho(\mathbf{x}, t)$ gives

$$\begin{aligned} \nabla^2\psi - 2\mathrm{i}a_0\frac{v_5}{v_4}\partial_t\psi + \mathrm{i}k(\nabla\cdot\nabla\rho)\psi + 2\mathrm{i}k\nabla\rho\cdot\nabla\psi - \\ -2\mathrm{i}k\frac{v_5}{v_4}\,a_0(\partial_t\rho)\psi - k^2(\nabla\rho\cdot\nabla\rho)\psi = 0\,. \end{aligned} \tag{65}$$

With $v_4 = v_5$, $a_0 = -m$ and $k = m$, this gives

$$\mathrm{i}\partial_t\psi = m(\partial_t\rho)\psi - \frac{1}{2m}\left(\nabla + \mathrm{i}m\nabla\rho\right)^2\psi\,, \tag{66}$$

which is equation (6.5) of [7].

Now consider a Galilei–null vector

$$\tilde{u}_p^\mu \equiv \left(\mathbf{u}_p, v_4, \frac{1}{2v_5}\mathbf{u}_p^2\right), \tag{67}$$

and a real scalar field $\tilde{\phi}$ with Eq. (9). Take the scalar product

$$\tilde{\Phi}_p = \tilde{u}_p^\mu\partial_\mu\tilde{\phi} = \mathbf{u}_p\cdot\nabla\phi + \tfrac{1}{2}a_0\mathbf{u}_p^2 + \partial_t\phi\,, \tag{68}$$

which, using $v_4 = v_5$ and $a_0 = -1$, leads to

$$\Phi_p = \partial_t\phi + \mathbf{u}_p\cdot\nabla\phi - \tfrac{1}{2}\mathbf{u}_p^2\,. \tag{69}$$

Similarly, let us construct $\Phi_m = \partial_t\phi + \mathbf{u}_m\cdot\nabla\phi - \frac{1}{2}\mathbf{u}_m^2$. The subscripts p and m refer to the collective and individual modes of the fluid, respectively. Next, consider

$$\begin{aligned} \chi_m &\equiv \frac{\mathrm{i}}{2}\,\tilde{u}_m^\mu\left[\tilde{\psi}^*\partial_\mu\tilde{\psi} - (\partial_\mu\tilde{\psi}^*)\tilde{\psi}\right] = \\ &= \frac{\mathrm{i}}{2}\left(\psi^*\left[\partial_t\psi + \mathbf{u}_m\cdot\nabla\psi\right] - \left[\partial_t\psi^* + \mathbf{u}_m\cdot\nabla\psi^*\right]\psi + \mathrm{i}\mathbf{u}_m^2 a_0\psi^*\psi\right), \end{aligned} \tag{70}$$

where we have defined the embedding as in Eqs. (8) and (10). If we choose $a_0 = 0$, then the last term in χ_m vanishes:

$$\chi_m = \psi^*\left[\partial_t\psi + \mathbf{u}_m\cdot\nabla\psi\right] - \left[\partial_t\psi^* + \mathbf{u}_m\cdot\nabla\psi^*\right]\psi\,. \tag{71}$$

If we use the definitions

$$\eta_m(x) \equiv \alpha\psi^*(x) + \alpha^*\psi(x)\,, \tag{72}$$

and

$$\rho_m(x) \equiv m\psi^*(x)\psi(x)\,, \tag{73}$$

then the Lagrangian of equation (3.12) of [7] takes the form

$$\mathcal{L} = \frac{\rho_0}{2v_0^2}(\Phi_p(x) - v_0^2)^2 + \eta_m(x)\Phi_p(x) + \rho_m(x)(\Phi_p(x) - \Phi_m(x)) + \chi_m(x)\,, \tag{74}$$

where χ_m, η_m and ρ_m are given by Eqs. (71), (72) and (73), respectively, and Φ_p, Φ_m are given by Eq. (69).

5.2 Generalized models for non−barotropic fluids

In this section, we generalize Eq. (59) by relaxing the condition $p \propto \rho^2$ (p: pressure, ρ: its density) to allow $p \propto \rho^\gamma$ ($\gamma \geq 1$). For $\gamma \neq 1$, let us consider

$$\tilde{\mathcal{L}} = k_p(\partial\tilde{\phi}\partial\tilde{\phi} - v_0^2)^p, \tag{75}$$

for which the equation of motion reads

$$\left(\tfrac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - v_0^2\right)\partial_\nu\partial^\nu\tilde{\phi} + (p-1)\partial_{\mu\nu}\tilde{\phi}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi} = 0\,. \tag{76}$$

When we consider Eq.(8) with $v_4 = v_5$ and $a_0 = -1$, we find

$$v_0^2\nabla^2\phi - (p-1)\partial_t^2\phi = \nabla^2\phi\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi\right) + (p-1)\nabla\phi\cdot\nabla\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + 2\partial_t\phi\right). \tag{77}$$

For $p = 1$, we obtain

$$v_0^2\nabla^2\phi = \nabla^2\phi\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi\right)\,, \tag{78}$$

whereas the case $p \neq 1$ is equivalent to the Takahashi model. Another possibility, corresponding to $\gamma = 1$, is

$$\tilde{\mathcal{L}}[\tilde{\phi}] = k\exp\left(\partial\tilde{\phi}\partial\tilde{\phi} - v_0^2\right). \tag{79}$$

The corresponding equation of motion,

$$v_0^2\partial_\mu\partial^\mu\tilde{\phi} + \partial_{\mu\nu}\tilde{\phi}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi} = 0\,, \tag{80}$$

reduces to

$$v_0^2\nabla^2\phi + \partial_{tt}\phi + \nabla\phi\cdot\nabla\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + 2\partial_t\phi\right) = 0\,. \tag{81}$$

Other equations relevant in condensed matter physics may be obtained with

$$\tilde{\mathcal{L}}[\tilde{\psi},\tilde{\psi}^*] \propto (\partial\tilde{\psi}\partial\tilde{\psi}^* - V(|\tilde{\psi}|))^p, \tag{82}$$

for a *complex* field $\tilde{\psi}$. The choice $p = 1$ and $V = \lambda|\tilde{\psi}|^4$, with the embedding in Eqs. (8) and (10), gives us

$$\mathcal{L} = k_1\left(\nabla\psi\cdot\nabla\psi^* - \mathrm{i}m(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - \lambda|\psi|^4\right). \tag{83}$$

The Euler–Lagrange equation, with $v_4 = v_5$ and $a_0 = -m$ leads to the non-linear Schrödinger equation,

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + \frac{\lambda}{m}|\psi|^2\psi\,. \tag{84}$$

Similar equations are used in effective theories of superconductivity and Bose–Einstein condensation (see references in [3]).

5.3 Model of non-viscous fluids and liquid helium

Consider Eq. (49), with Clebsch–like transformation $\partial\tilde{\phi} \to \partial\tilde{\phi} + \tilde{\alpha}\partial\tilde{\beta}$,

$$\tilde{\mathcal{L}} = -\frac{\tilde{\rho}}{2v_0^2}(\partial_\mu\tilde{\phi} + \tilde{\alpha}\partial_\mu\tilde{\beta})(\partial^\mu\tilde{\phi} + \tilde{\alpha}\partial^\mu\tilde{\beta}) - V(\tilde{\rho})\,. \tag{85}$$

If we define

$$\begin{aligned} \tilde{\alpha}(x) &= \alpha(\mathbf{x}, t)\,, \\ \tilde{\beta}(x) &= \beta(\mathbf{x}, t)\,, \\ \tilde{\rho}(x) &= \rho(\mathbf{x}, t)\,, \end{aligned} \tag{86}$$

together with Eq. (9) for $\tilde{\phi}(x)$, and Eq. (8) for space–time (here we take $a_0 = +1$), then the Lagrangian of Eq. (85) on the Newtonian space–time becomes

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi - \frac{1}{2}\nabla\phi\cdot\nabla\phi + \alpha(\partial_t\beta - \frac{1}{2}\alpha\nabla\beta\cdot\nabla\beta) - \alpha\nabla\phi\cdot\nabla\beta\right) - V(\rho)\,. \tag{87}$$

This may be expressed as

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi + \alpha\partial_t\beta - \frac{1}{2}\mathbf{v}^2\right) - V(\rho)\,, \tag{88}$$

where $\mathbf{v} = -\nabla\phi - \alpha\nabla\beta$. This Lagrangian was employed by Thellung and Ziman (see Section 4.3 of Ref. [3]).

6 Interacting Fermi field

The Galilean version of the Dirac equation has been investigated using the present formalism in Ref. [2]. Therein we have retrieved the Lévy-Leblond equations [11], as well as the Pauli equation, spin–orbit interaction and a Darwin–like term. Moreover, a generalized model involving the interaction of a non-abelian gauge field with the Dirac field has been presented. Hereafter we complete the discussion by examining the related Lagrangian densities.

First, let us consider the Galilean Dirac Lagrangian for the free Fermi field:

$$\mathcal{L}_{\mathrm{GDirac}} = \overline{\Psi}\left(\mathrm{i}\gamma^\mu\overleftrightarrow{\partial}_\mu - k\right)\Psi\,, \tag{89}$$

where $A\overleftrightarrow{\partial}B \equiv \frac{1}{2}[A\partial B - (\partial A)B]$. We use the following gamma matrices (see Omote et al. [7]):

$$\gamma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{90}$$

where each entry is a two-by-two matrix and the σ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{91}$$

These gamma matrices satisfy the usual relation: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{\mu\nu}$ is the Galilean metric. Following Omote et al. [7], we define the adjoint spinor by $\overline{\Psi} = \Psi^\dagger\zeta$, with

$$\zeta = \frac{-\mathrm{i}}{\sqrt{2}}\left(\gamma^4 + \gamma^5\right) = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{92}$$

Let us now utilize the embedding within the Lagrangian (89). From the definitions above, with spinor $\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and Eq. (10), we find that Eq. (89) becomes

$$\begin{aligned} \mathcal{L}_{\text{GDirac}} = \tfrac{1}{2}\Big[&(\nabla\psi_2^\dagger)\cdot\sigma\psi_1 - \psi_2^\dagger\sigma\cdot\nabla\psi_1 - \psi_1^\dagger\sigma\cdot\nabla\psi_2 + (\nabla\psi_1^\dagger)\cdot\sigma\psi_2\Big] - \\ &-\tfrac{\sqrt{2}}{2}(\psi_1^\dagger\partial_t\psi_1 - (\partial_t\psi_1^\dagger)\psi_1) + \mathrm{i}m\sqrt{2}\psi_2^\dagger\psi_2 - \mathrm{i}k(\psi_2^\dagger\psi_1 - \psi_1^\dagger\psi_2)\,. \end{aligned} \tag{93}$$

Variations of this Lagrangian with respect to ψ_1 and ψ_2 lead to

$$\mathrm{i}\sqrt{2}\partial_t\psi_1^\dagger + \mathrm{i}\nabla\psi_2^\dagger\cdot\sigma + k\psi_2^\dagger = 0\,, \tag{94}$$

and

$$(\nabla\psi_1^\dagger)\cdot\sigma + \mathrm{i}k\psi_1^\dagger + \mathrm{i}\sqrt{2}m\psi_2^\dagger = 0\,, \tag{95}$$

respectively. With respect to their conjugates $\psi_1^\dagger$ and $\psi_2^\dagger$, we obtain

$$\mathrm{i}\sqrt{2}\partial_t\psi_1 + (\mathrm{i}\sigma\cdot\nabla + k)\psi_2 = 0\,, \tag{96}$$

and

$$(\mathrm{i}\sigma\cdot\nabla\psi_1 - k)\psi_1 + \sqrt{2}m\psi_2 = 0\,, \tag{97}$$

respectively. When we substitute ψ_2 from Eq. (96) into Eq. (97), we obtain

$$\mathrm{i}\partial_t\psi_1 = -\frac{1}{2m}\left(\nabla^2 + k^2\right)\psi_1\,. \tag{98}$$

If we absorb the constant k into the energy operator, we clearly obtain Eq. (15) with a constant potential.

Now let us return to the Lagrangian of Eq. (89) and find the Euler–Lagrange equations before performing any embedding. The variation of $\mathcal{L}_{\text{GDirac}}$ with respect to $\overline{\Psi}$ gives

$$(\mathrm{i}\gamma^\mu\partial_\mu - k)\Psi = 0\,. \tag{99}$$

The Euler–Lagrange equation with respect to its adjoint gives

$$\overline{\Psi}\big(\mathrm{i}\gamma^\mu \overleftarrow{\partial}_\mu + k\big) = 0\,, \tag{100}$$

where $A\overleftarrow{\partial} \equiv \partial A$. Now, by using the embedding defined in Eq. (10) into Eq. 99) leads to Eqs. (96) and (97), whereas with Eq. (100) we retrieve Eqs. (94) and (95). As usual, by multiplying Eq. (99) on the left with $\mathrm{i}\gamma^\mu\partial_\mu + k$, we find Eq. (14) with a constant potential.

Now we consider the interaction of the Fermi field with the gauge field. Here we consider the Galilean covariant version of the QED Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\mathrm{GQED}} &= \mathcal{L}_{\mathrm{GDirac}} + \mathcal{L}_{\mathrm{int.}} + \mathcal{L}_{\mathrm{GEM.}}\,, \\
&= \overline{\Psi}(\mathrm{i}\gamma^\mu \overleftrightarrow{\partial}_\mu - k)\Psi - e\overline{\Psi}\gamma^\mu\Psi A_\mu - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \\
&= \overline{\Psi}(\mathrm{i}\gamma^\mu \overleftrightarrow{D}_\mu - k)\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\,,
\end{aligned} \tag{101}$$

with the usual definition

$$D_\mu \equiv \partial_\mu + \mathrm{i}eA_\mu\,. \tag{102}$$

In order to expand this Lagrangian in terms of the embedding in Eq. (8), we make use of earlier results, namely Eqs. (10), (30), (34), and (93), and we get

$$\begin{aligned}
\mathcal{L}_{\mathrm{GQED}} = {} & \tfrac{1}{2}\Big[(\nabla\psi_2^\dagger)\cdot\sigma\psi_1 - \psi_2^\dagger\sigma\cdot\nabla\psi_1 - \psi_1^\dagger\sigma\cdot\nabla\psi_2 + (\nabla\psi_1^\dagger)\cdot\sigma\psi_2\Big] - \\
& -\tfrac{\sqrt{2}}{2}\big(\psi_1^\dagger\partial_t\psi_1 - (\partial_t\psi_1^\dagger)\psi_1\big) + \mathrm{i}m\sqrt{2}\psi_2^\dagger\psi_2 - \mathrm{i}k(\psi_2^\dagger\psi_1 - \psi_1^\dagger\psi_2) - \\
& -\mathrm{i}e(\psi_1^\dagger\sigma\cdot\mathbf{A}\psi_2 + \psi_2^\dagger\sigma\cdot\mathbf{A}\psi_1) + \mathrm{i}e\sqrt{2}(\phi_m\psi_1^\dagger\psi_1 + \phi_e\psi_2^\dagger\psi_2) - \\
& -\tfrac{1}{2}(\nabla\times\mathbf{A})^2 + (\nabla\phi_m + \partial_t\mathbf{A})\cdot\nabla\phi_e + \tfrac{1}{2}(\partial_t\phi_e)^2\,.
\end{aligned} \tag{103}$$

Note that we take $c = 1$. The Euler–Lagrange equation with respect to $\psi_1^\dagger$ leads to

$$\sigma\cdot(\mathrm{i}\nabla - e\mathbf{A})\psi_2 + \sqrt{2}(\mathrm{i}\partial_t + e\phi_m)\psi_1 + k\psi_2 = 0\,, \tag{104}$$

and to

$$\sigma\cdot(\mathrm{i}\nabla - e\mathbf{A})\psi_1 - k\psi_1 + \sqrt{2}(m + e\phi_e)\psi_2 = 0\,, \tag{105}$$

when it is calculated with respect to $\psi_2^\dagger$. This leads to the Lévy-Leblond Eqs. [11] if we take $k = 0$ and choose the magnetic limit, defined in Section 3, by taking $\phi_e = 0$.

Now we briefly discuss the opposite procedure, which consists in considering the Euler–Lagrange equations in the $(4,1)$ manifold, and then defining the embedding. From the variation of $\mathcal{L}_{\mathrm{GQED}}$ with $\overline{\Psi}$, we find

$$(\mathrm{i}\gamma^\mu D_\mu - k)\Psi = (\mathrm{i}\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - k)\Psi = 0\,. \tag{106}$$

The field equations of motion with respect to A_μ read $\partial_\mu F^{\mu\nu} = e\overline{\Psi}\gamma^\nu\Psi$. The covariant expansion of these equations is discussed in Section 4 of Ref. [2]. Therein, it was shown to lead to the Pauli equation, and to describe the correct Landé factor of the electron's intrinsic magnetic moment, as well as the spin–orbit coupling and a term similar to the Darwin term.

7 Bhabha−DKP equation: spin zero and spin one

This section is a summary of Refs. [5, 6]. The Duffin–Kemmer–Petiau (DKP) equation is

$$(\beta^\mu\partial_\mu + k)\Psi = 0\,, \tag{107}$$

and the matrices β obey the so-called DKP algebra:

$$\beta^\mu \beta^\lambda \beta^\nu + \beta^\nu \beta^\lambda \beta^\mu = g^{\mu\lambda} \beta^\nu + g^{\nu\lambda} \beta^\mu \,, \tag{108}$$

where $g^{\mu\nu}$ is the Galilean metric. The adjoint of Ψ is defined by $\overline{\Psi} \equiv \Psi^\dagger \eta$, where

$$\eta = (\beta^4 + \beta^5)^2 + \mathbf{1}. \tag{109}$$

In the following, we use the momentum version of Eq. (107):

$$(\beta^\mu p_\mu - \mathrm{i}k)\Psi = 0 \,. \tag{110}$$

7.1 DKP equation for spin zero

For spinless particles, we take

$$\beta^1 = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}, \quad \beta^2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix},$$
$$\beta^3 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}, \quad \beta^4 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&-1&0 \end{pmatrix}, \tag{111}$$
$$\beta^5 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&-1&0&0 \end{pmatrix}.$$

They belong to the Lie algebra $so(5,1)$, with $J^{\mu 6} \equiv \beta^\mu$, $\mu = 1, \ldots, 5$. Matrices in Eq. (111) satisfy the DKP algebra given by Eq. (108).

Hereafter, we review the results obtained in [6]. If we introduce a DKP spinor

$$\Psi \equiv \begin{pmatrix} \mathbf{A} \\ \theta \\ \varphi \\ \phi \end{pmatrix}, \tag{112}$$

with $\mathbf{A} = (A_x, A_y, A_z)$, then Eq. (110) gives

$$\begin{aligned} &-\mathrm{i}k\mathbf{A} + \mathbf{p}\phi = \mathbf{0}\,, \\ &-\mathrm{i}k\theta + p_4\phi = 0\,, \\ &-\mathrm{i}k\varphi + p_5\phi = 0\,, \\ &\mathbf{p}\cdot\mathbf{A} - p_5\theta - p_4\varphi - \mathrm{i}k\phi = 0\,. \end{aligned} \tag{113}$$

These equations can be expressed in terms of ϕ as $\mathbf{p}^2\phi - 2p_4p_5\phi + k^2\phi = 0$, and this becomes the Schrödinger equation,

$$E\phi = \frac{\mathbf{p}^2}{2m}\,\phi\,, \tag{114}$$

by defining $\mathbf{p} \to (\mathbf{p}, p_4, p_5)$ such that $p_4p_5 = mE$ and by absorbing the constant k into the energy as $E \to E - k^2/(2m)$.

The harmonic oscillator is described by performing the non-minimal substitution $\mathbf{p} \to \mathbf{p} + \mathrm{i}\omega\eta\mathbf{r}$, with η given by Eqs. (109) and (111). After performing the change $\omega \to m\omega$, rather than Eq. (114), we find

$$E\phi = \left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}\,m\omega^2\mathbf{r}^2 - \frac{3}{2}\,\hbar\omega\right)\phi\,. \tag{115}$$

This equation has been obtained in Ref. [10] as a low–velocity limit of the corresponding relativistic problem.

7.2 DKP equation for spin one

If we use the shorthand notation e_{ij} to represent a fifteen-by-fifteen matrix whose only non-zero entry is ij, defined to be one, that is, $(e_{ij})_{mn} \equiv \delta_{im}\delta_{jn}$, then the DKP generators of the spin one representation are

$$\begin{aligned} \beta^1 &= e_{13,1} + e_{14,4} + e_{12,8} - e_{11,9} - e_{9,11} + e_{8,12} + e_{1,13} + e_{4,14}\,, \\ \beta^2 &= e_{13,2} + e_{14,5} - e_{12,7} + e_{10,9} + e_{9,10} - e_{7,12} + e_{2,13} + e_{5,14}\,, \\ \beta^3 &= e_{13,3} + e_{14,6} + e_{11,7} - e_{10,8} - e_{8,10} + e_{7,11} + e_{3,13} + e_{6,14}\,, \\ \beta^4 &= -e_{10,4} - e_{11,5} - e_{12,6} + e_{1,10} + e_{2,11} + e_{3,12} + e_{15,14} + e_{13,15}\,, \\ \beta^5 &= -e_{10,1} - e_{11,2} - e_{12,3} + e_{4,10} + e_{5,11} + e_{6,12} - e_{15,13} - e_{14,15}\,. \end{aligned} \tag{116}$$

They also correspond to the basis elements $J^{\mu 6}$ of $so(5,1)$.

Here we only consider the DKP simple harmonic oscillator by first performing the non-minimal substitution of the previous section, where η is given by Eqs. (109) and (116), and substitute Eq. (116) into Eq. (110), with the DKP spinor given by

$$\Psi = \begin{pmatrix} v_1 \\ \vdots \\ v_{15} \end{pmatrix}. \tag{117}$$

Then we obtain

$$E\mathbf{A} = \left[\frac{\mathbf{p}^2}{2m} + \frac{1}{2}\, m\omega^2\mathbf{r}^2 - \frac{3}{2}\,\hbar\omega - \frac{\omega}{\hbar}\,\mathbf{L}\cdot\mathbf{S}\right]\mathbf{A}\,. \tag{118}$$

This is the non-relativistic version obtained earlier [Equation (16) in Ref. [10]]. It should be emphasized that both spin 0 and spin 1 require the same definitions of p_4 and p_5 as for spinless particles.

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