On conservation laws for the potential Zabolotskaya–Khokhlov equation

V. Rosenhaus

Department of Mathematics and Statistics, California State University, Chico, CA 95929, USA

We study local conservation laws for the potential Zabolotskaya–Khokhlov equation in three–dimensional case. We analyze an infinite Lie point symmetry group of the equation, and generate a finite number of conserved quantities corresponding to infinite symmetries through appropriate boundary conditions.

PACS: 11.30.-j, 02.20.Tw
Key words: infinite symmetries, conservation laws

The Zabolotskaya–Khokhlov equation in three dimensions [1]

$$v_{xt} - v_x^2 - v v_{xx} - v_{yy} - v_{zz} = 0$$

describes the propagation of a confined three–dimensional sound beam in a slightly nonlinear medium without absorption or dispersion. Classical symmetries of the Zabolotskaya–Khokhlov (Z-K) equation and some invariant solutions were found in [2], and [3], (see also [4]). Similarity solutions of the Z-K equation (in two dimensions) were obtained in [5], and [6]. It was shown in [7] that the Zabolotskaya–Khokhlov equation in two dimensions is invariant under the same group of Lie point transformations as the Kadomtsev–Pertviashvili equation.

Introducing $v = u_x$ we will get the potential Zabolotskaya–Khokhlov equation

$$u_{xt} - u_x u_{xx} - u_{yy} - u_{zz} = 0\,. \tag{1}$$

The potential Zabolotskaya–Khokhlov equation can be obtained from the equation of non-stationary transonic gas flows for three–dimensional non-steady motion in a compressible fluid

$$2u_{xt} + u_x u_{xx} - u_{yy} - u_{zz} = 0\,, \tag{2}$$

where $u = u(x, y, z, t)$ by the following transformation: $x \to -x$, $t \to -2t$. The classical Lie point symmetry group of the equation (2) was studied in [8] and [9]; see also [10]. The basis of conservation laws (in two dimensions) without regard to boundary conditions has been studied in [11].

In the present paper we will analyze essential conservation laws of potential Zabolotskaya–Khokhlov equation (1). Following the ideology of the approach proposed in [12, 13], we construct a finite set of conserved quantities for the equation (1) associated with its infinite symmetry subgroups. Each conservation law corresponds to a specific choice of an arbitrary function in the group generators, which determines the appropriate boundary condition.

By a conservation law for a differential equation $\omega(x, u, u_i, \ldots) = 0$ is meant a continuity equation

$$D_\mu K_\mu \doteq 0\,, \quad \mu = 1, \ldots, n\,; \quad n = 4\,,$$

which is satisfied for any solutions of the equation $\omega(x, u, u_i, \ldots) = 0$. We will exclude from consideration trivial conservation laws [14], for which the components of the vector K_μ vanish on the solutions: $K_\mu \doteq 0$ ($\mu = 1, \ldots, n$), or the continuity equation is satisfied in the whole space: $D_\mu K_\mu = 0$. Let us further reduce the set of conservation laws to essential conservation laws. By an **essential** conservation law, we will mean such non-trivial conservation law $D_\mu K_\mu \doteq 0$, which gives rise to a non-vanishing conserved quantity

$$D_t \iiint K_t \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \doteq 0\,, \quad K_t \dot{\neq} 0\,. \tag{3}$$

Let

$$S = \int L(x^i, u^a, u^a_i, \cdots)\, \mathrm{d}^n x$$

be the action functional, where L is the Lagrangian density, $x^i = (x, y, \ldots, t)$, $i = 1, \ldots, m+1$ are independent variables and u^a, $a = 1, \ldots, n$ are dependent variables; $u^a_i \equiv \partial u^a / \partial x^i$, $u^a_{ij} \equiv \partial^2 u^a / \partial x^i \partial x^j$. Then the equations of motion are

$$E^a(L) \equiv \omega^a(x, u, u_i, u_{ij}, \ldots) = 0\,, \quad a = 1, \ldots, n\,, \tag{4}$$

where E is the Euler–Lagrange operator

$$E^a = \frac{\partial}{\partial u^a} - D_i \frac{\partial}{\partial u^a_i} + \sum_{j \geq i} D_i D_j \frac{\partial}{\partial u^a_{ij}} + \cdots\,. \tag{5}$$

Consider an infinitesimal transformation with the canonical operator

$$X_\alpha = \alpha^a \frac{\partial}{\partial u^a} + \sum_{i=1}^{m+1} (D_i \alpha^a) \frac{\partial}{\partial u^a_i} + \sum_{1 \leq i \leq j \leq m+1} (D_i D_j \alpha^a) \frac{\partial}{\partial u^a_{ij}} + \cdots\,, \tag{6}$$

where $\alpha^a = \alpha^a(x, u, u_i, \ldots)$. Variation of the functional S under the transformation with operator X_α is

$$\delta S = \int X_\alpha L\, \mathrm{d}^n x\,. \tag{7}$$

Let us consider a variational (Noether) symmetry X_α

$$X_\alpha L = D_i M_i\,, \tag{8}$$

where $M_i(x)$ are smooth functions. In the future we will use the Noether identity (see e.g. [13]):

$$X_\alpha = \alpha^a E^a + \sum_{i=1} D_i R_{\alpha i}\,, \tag{9}$$

where

$$R_{\alpha i} = \alpha^a \frac{\partial}{\partial u_i^a} + \left\{ \sum_{k\geq i} (D_k \alpha^a) - \alpha^a \sum_{k\leq i} D_k \right\} \frac{\partial}{\partial u_{ik}^a} + \dots \,. \tag{10}$$

Applying the Noether identity (9) to L, and combining with (8) and (10) we will obtain

$$D_i(M_i - R_{\alpha i} L) = \alpha^a \omega^a \,. \tag{11}$$

The equation (11) applied on the solution manifold ($\omega = 0$, $D_i\omega = 0, \dots$) leads to a continuity equation

$$D_i(M_i - R_{\alpha i} L) \doteq 0 \,. \tag{12}$$

Thus, any finite variational symmetry transformation X_α (8) leads to a conservation law (12) (the First Noether Theorem) with the characteristic α.

The Second Noether Theorem [15] deals with a case of an infinite variational symmetry group where the symmetry vector α is of the form

$$\alpha = ap(x) + b_i D_i p(x) + c_{ij} D_i D_j p(x) + \dots \,, \tag{13}$$

and $p(x)$ is an arbitrary function of all base variables of the space. A general situation when $p(x)$ is an arbitrary function of not all base variables was analyzed in [13]. In this paper, we will be mostly interested in the case of arbitrary functions of time t. Consider a variational (Noether) symmetry α of the form

$$\alpha = a\gamma(t) + b\gamma'(t) + c\gamma''(t) + \dots + h\gamma^{(l)}(t) \tag{14}$$

for a differential equation with the Lagrangian function $L = L(x^i, u, u_i)$, $x^i = (x^1, x^2, \dots, x^m, t)$. For a Noether symmetry transformation X_α we have

$$\delta S = \iint \delta L \, \mathrm{d}^{m+1}x = \iint X_\alpha L \, \mathrm{d}^{m+1}x = \iint D_i M_i \, \mathrm{d}^{m+1}x = 0 \,. \tag{15}$$

Therefore, the following conditions for M_i (Noether boundary conditions) should be satisfied

$$M_i(x, u, \dots)\Big|_{x^i \to \partial D} = 0 \,, \quad \forall i = 1, \dots, m+1 \,. \tag{16}$$

Equations (16) are usually satisfied for a "regular" asymptotic behavior, u, $u_i \to 0$ as $x \to \pm\infty$, or for periodic solutions. Let us consider now another type of boundary conditions related to the existence of local conserved quantities. Integrating equation (12) over the space $(x^1, x^2, \dots, x^m)$ and restricting ourselves to the solution manifold, we get

$$\iint \mathrm{d}x^1 \mathrm{d}x^2 \dots \mathrm{d}x^m D_t(M_t - R_{\alpha t} L) \doteq \iint \mathrm{d}x^1 \dots \mathrm{d}x^m \sum_{i=1}^{m} D_i(R_{\alpha i} L - M_i) \,. \tag{17}$$

Applying the Noether boundary condition (16) and requiring the LHS of (17) to vanish on the solution manifold we obtain the "strict" boundary conditions [13]

$$R_{\alpha 1} L\Big|_{x^1 \to \partial D} = R_{\alpha 2} L\Big|_{x^2 \to \partial D} = \dots = R_{\alpha m} L\Big|_{x^m \to \partial D} = 0 \,. \tag{18}$$

In the case $L = L(x, u, u_j)$, strict boundary conditions (18) take a simple form

$$\alpha \frac{\partial L}{\partial u_i}\Big|_{x^i \to \partial D} = 0\,, \quad \forall i = 1, \ldots, m\,. \tag{19}$$

In order for the system to possess local conserved quantities, both Noether (16) and strict boundary conditions (18) have to be satisfied. In this case corresponding Noether conservation law can be found in the form

$$\iint \mathrm{d}x^1 \mathrm{d}x^2 \ldots \mathrm{d}x^m\, D_t(M_t - R_{\alpha t} L) \doteq 0\,. \tag{20}$$

Let us write M_t as

$$M_t = A\gamma(t) + B\gamma'(t) + C\gamma''(t) + \ldots + H\gamma^{(l)}(t)\,. \tag{21}$$

Using (20), we obtain

$$D_t \iint \mathrm{d}x^1 \mathrm{d}x^2 \ldots \mathrm{d}x^m \left[\gamma(t)A_1 + \gamma'(t)A_2 + \ldots + \gamma^{(l)}(t)A_l\right] \doteq 0\,. \tag{22}$$

where

$$A_1 = \left(A - a\,\frac{\partial L}{\partial u_t}\right), \quad A_2 = \left(B - b\,\frac{\partial L}{\partial u_t}\right), \quad \ldots, \quad A_l = \left(H - h\,\frac{\partial L}{\partial u_t}\right).$$

Since $\gamma(t)$ is an arbitrary function we get

$$\int \mathrm{d}x^1 \mathrm{d}x^2 \ldots \mathrm{d}x^m \left\{A - a\,\frac{\partial L}{\partial u_t}\right\} \doteq \int \mathrm{d}x^1 \mathrm{d}x^2 \ldots \mathrm{d}x^m \left\{B - b\,\frac{\partial L}{\partial u_t}\right\} \doteq \ldots$$
$$\ldots \doteq \int \mathrm{d}x^1 \mathrm{d}x^2 \ldots \mathrm{d}x^m \left\{H - h\,\frac{\partial L}{\partial u_t}\right\} \doteq 0\,. \tag{23}$$

Obviously, equations (23), in general, do not determine a system of conservation laws but additional constraints. Thus, Noether symmetries (empowered by strict boundary conditions) with arbitrary functions of time instead of conservation laws lead to a set of additional constraints imposed on the function u and its derivatives. Therefore, the satisfaction of the strict boundary conditions (18), along with the Noether boundary condition (16), becomes critical in the sense of avoiding additional constraints (23). Correspondingly, we have three possible situations:

1) Strict boundary conditions (18) (or (19)), along with the Noether boundary condition (16), can be satisfied for an arbitrary function $\gamma(t)$. Then the system (23), as a consequence of an infinite symmetry (14), provides additional constraints that the function u and its derivatives must satisfy. No conservation laws are associated with the symmetry. This situation materializes for the cases when the system (23) is not overly restrictive.

2) Strict boundary conditions (18) along with the Noether boundary condition (16), can be satisfied for some particular functions $\gamma(t)$. In this case the (finite)

symmetry (14) will lead to the Noether conservation law (20) in agreement with the First Noether Theorem. Additional constraints (23) do not appear.

3) Strict boundary conditions cannot be satisfied for any functions $\gamma(t)$. In this case a consequence of an infinite symmetry will be the fact that the solutions of the original differential equation with the boundary conditions (18), (16) do not exist. Therefore, in order to avoid additional constraints (23) we have to find some particular functions $\gamma(t)$ (if they exist) leading to different boundary conditions than the ones in general case [13] (when function $\gamma(t)$ is arbitrary). Each choice of such a function $\gamma(t)$ gives rise to a respective conserved quantity.

Let us apply the approach above for finding non-vanishing conserved densities of the potential Zabolotskaya–Khokhlov equation (1) with boundary conditions on infinity

$$u_{xt} - u_x u_{xx} - u_{yy} - u_{zz} = 0\,.$$

The Lagrangian function of the equation is

$$L = -\frac{u_x u_t}{2} + \frac{u_x^3}{6} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\,. \tag{24}$$

The Lie point symmetry group of the equation is formed by the following operators:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}\,, \qquad X_2 = 5t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} + 3y\,\frac{\partial}{\partial y} + 3z\,\frac{\partial}{\partial z} - 3u\,\frac{\partial}{\partial u}\,,\\
X_3 &= t\,\frac{\partial}{\partial t} - x\,\frac{\partial}{\partial x} - 3u\,\frac{\partial}{\partial u}\,, \qquad X_4 = z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z}\,,\\
X_5 &= -5t^2\,\frac{\partial}{\partial t} - \left[2tx + \tfrac{3}{2}\,(y^2+z^2)\right]\frac{\partial}{\partial x} - 6ty\,\frac{\partial}{\partial y} - 6tz\,\frac{\partial}{\partial z} + (x^2+6tu)\,\frac{\partial}{\partial u}\,,\\
X_\gamma &= -\gamma(t)\,\frac{\partial}{\partial x} + \left[\gamma'(t)x + \tfrac{1}{4}(y^2+z^2)\gamma''(t)\right]\frac{\partial}{\partial u}\,,\\
X_f &= 2f\,\frac{\partial}{\partial y} + yf'\,\frac{\partial}{\partial x} - \left[xyf'' + \tfrac{1}{4}f'''\left(\tfrac{1}{3}\,y^3 + z^2 y\right)\right]\frac{\partial}{\partial u}\,,\\
X_h &= 2h\,\frac{\partial}{\partial z} + zh'\,\frac{\partial}{\partial x} - \left[xzh'' + \tfrac{1}{4}\,h'''\left(\tfrac{1}{3}\,z^3 + zy^2\right)\right]\frac{\partial}{\partial u}\,,
\end{aligned} \tag{25}$$

where $\gamma(t)$, $f(t)$, $h(t)$ are arbitrary functions. Let us analyze infinite subgroups of the symmetry group (25).

$$X_\gamma = -\gamma(t)\,\frac{\partial}{\partial x} + \left[\gamma'(t)x + \tfrac{1}{4}\left(y^2+z^2\right)\gamma''(t)\right]\frac{\partial}{\partial u}\,, \tag{26}$$

$\gamma(t)$ is arbitrary. We have

$$\begin{gathered}
\alpha = \gamma' x + \tfrac{1}{4}\left(y^2+z^2\right)\gamma'' + \gamma u_x\,, \quad X_\alpha L = D_i M_i\,, \quad M_t = -\tfrac{1}{2}\,u\gamma'\,.\\
M_x = \gamma L - \tfrac{1}{2}\,xu\gamma'' - \tfrac{1}{8}\left(y^2+z^2\right)u\gamma'''\,, \quad M_y = \tfrac{1}{2}\,yu\gamma''\,, \quad M_z = \tfrac{1}{2}\,zu\gamma''\,,\\
a = u_x\,, \quad b = x\,, \quad c = \tfrac{1}{4}\left(y^2+z^2\right)), \quad A = C = 0\,, \quad B = -\tfrac{1}{2}\,u\,.
\end{gathered} \tag{27}$$

The form of Noether and strict boundary conditions in this case depends on the function $\gamma(t)$.

A. $\gamma(t)$ *is arbitrary*

Noether conditions (16) for X_α are

$$M_i(x,u,\ldots)\Big|_{x^i\to\partial D} = 0\,, \quad \forall i = 1,\ldots,4\,,$$

or

$$u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad xu \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad yu \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad zu \underset{z\to\pm\infty}{\longrightarrow} 0\,, \quad u \underset{t\to\pm\infty}{\longrightarrow} 0\,. \tag{28}$$

Strict conditions (19) take the form

$$xu_t,\, xu_x^2 \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad y^2u_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad z^2u_z \underset{y\to\pm\infty}{\longrightarrow} 0\,. \tag{29}$$

Thus, for the case of arbitrary $\gamma(t)$ we have the following boundary conditions:

$$xu,\, xu_x^2,\, u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad yu,\, y^2u_y, \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad zu,\, z^2u_z \underset{z\to\pm\infty}{\longrightarrow} 0\,, \quad u \underset{t\to\pm\infty}{\longrightarrow} 0\,. \tag{30}$$

No local conservation laws are associated with the Noether transformation X_α (26) for arbitrary $\gamma(t)$.

Let us consider now some specific forms of $\gamma(t)$ for which we can weaken our boundary conditions (28)−(29) in order to avoid restrictions (23).

B. $\gamma'(t) = 0$, $\gamma(t) = c = \text{const}$. Noether conditions are

$$u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,. \tag{31}$$

"Strict" boundary conditions in addition to (31) will have

$$u_xu_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u_xu_z \underset{z\to\pm\infty}{\longrightarrow} 0\,. \tag{32}$$

According to (20) we will get the following conservation law:

$$D_t \iint u_x^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \doteq 0\,, \tag{33}$$

or

$$\iint u_x^2 \mathrm{d}x\,\mathrm{d}y \doteq \text{const}\,.$$

(conservation of the x–component of momentum P_x). The corresponding continuity equation has the form

$$D_x\left(-\frac{u_x^3}{3} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\right) + D_y(-u_xu_y) + D_z(-u_xu_y) + D_t(\frac{u_x^2}{2}) \doteq 0\,. \tag{34}$$

C. $\gamma''(t) = 0$, $\gamma(t) = at$, $a = \text{const} \neq 0$. Noether conditions are

$$u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u \underset{t\to\pm\infty}{\longrightarrow} 0\,. \tag{35}$$

For strict boundary conditions we get

$$xu_t,\, xu_x^2 \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u_y,\, u_x u_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u_z,\, u_x u_z \underset{z\to\pm\infty}{\longrightarrow} 0\,. \tag{36}$$

The conservation law associated with the boundary conditions (35) – (36) takes the form

$$D_t \iint \left[xu_x - u + tu_x^2\right] \mathrm{d}x\, \mathrm{d}y \doteq 0\,. \tag{37}$$

As we can see, strict boundary conditions (36) together with Noether conditions (35) determined a nontrivial conservation law (37) with the characteristic $\alpha = x + tu_x$.

$$X_f = 2f\,\frac{\partial}{\partial y} + f'y\,\frac{\partial}{\partial x} - \left[xyf'' + \tfrac{1}{4}\,f'''\left(\tfrac{1}{3}\,y^3 + yz^2\right)\right]\frac{\partial}{\partial u}\,, \tag{38}$$

$f(t)$ is arbitrary function. For a corresponding canonical operator X_α we get

$$\alpha = 2fu_y + f'yu_x + f''xy + \tfrac{1}{4}\,f'''\left(\tfrac{1}{3}\,y^3 + yz^2\right). \tag{39}$$

Calculating $X_\alpha L$ we obtain

$$\begin{aligned} X_\alpha L &= D_i M_i\,, \\ M_x &= f'yL - \tfrac{1}{2}\,f'''xyu - \tfrac{1}{8}\,f^{(IV)}\left(\tfrac{1}{3}\,y^3 + yz^2\right)u\,, \\ M_y &= 2fL + f''xu + \tfrac{1}{4}\,f'''\left(y^2 + z^2\right)u\,, \\ M_z &= \tfrac{1}{2}\,f'''yzu\,, \\ M_t &= -\tfrac{1}{2}\,f''yu\,. \end{aligned} \tag{40}$$

In this case both strict and Noether boundary conditions are dependent on the function $f(t)$.

A. $f(t)$ is arbitrary.

From the Noether and strict boundary conditions we will get

$$xu,\, u_i,\, x^2u_x \underset{x\to\pm\infty}{\longrightarrow} 0, \quad u_i,\, y^2u,\, y^3u_y \underset{y\to\pm\infty}{\longrightarrow} 0, \quad zu,\, z^2u_z \underset{z\to\pm\infty}{\longrightarrow} 0, \quad u \underset{t\to\pm\infty}{\longrightarrow} 0. \tag{41}$$

No local conservation laws are associated with the Noether transformation X_α (38) with an arbitrary function $f(t)$. Let us consider now some specific forms of $f(t)$ for which we can weaken boundary conditions (41).

B. $f' = 0$, $f(t) = c = \text{const}$.

We have

$$M_x = 0\,, \quad M_y = cL\,, \quad M_z = M_t = 0\,, \quad \alpha = 2cu_y\,. \tag{42}$$

Noether conditions look as follows:

$$u_i \underset{y\to\pm\infty}{\longrightarrow} 0\,. \tag{43}$$

The strict boundary conditions have a form

$$u_t u_y,\ u_x^2 u_y \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u_y u_z \underset{z\to\pm\infty}{\longrightarrow} 0\,. \tag{44}$$

According to (20), the associated conservation law has a form:

$$D_t \iint u_x u_y \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \doteq 0 \tag{45}$$

and characteristic $\alpha = u_y$. Expression (45) is a conservation of the y–component of the momentum of the system P_y with regular boundary conditions (43) – (44).

C. $f'' = 0,\ f' \neq 0\ :\ f(t) = at,\ a = \mathrm{const} \neq 0.$
We have

$$M_x = ayL\,, \quad M_y = 2atL\,, \quad M_z = M_t = 0\,, \quad \alpha = 2atu_y + ayu_x\,. \tag{46}$$

Noether conditions are

$$u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u_i \underset{y\to\pm\infty}{\longrightarrow} 0\,, \tag{47}$$

and the strict boundary conditions (19) read

$$u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad yu_x u_y,\ u_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u_z \underset{z\to\pm\infty}{\longrightarrow} 0\,. \tag{48}$$

Boundary conditions (47) – (48) are weaker than the ones for the general case (41), and using (20) we obtain the following conserved quantity

$$D_t \iint \left[2tu_x u_y + yu_x^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \doteq 0\,. \tag{49}$$

D. $f''' = 0,\ f'' \neq 0\ :\ f(t) = \frac{1}{2}\,bt^2.$
We have

$$\begin{gathered} M_x = btyL\,, \quad M_y = bt^2L + bxu\,, \quad M_z = 0\,, \quad M_t = -\tfrac{1}{2}\,byu\,, \\ \alpha = bt^2u_y + btyu_x + bxy\,. \end{gathered} \tag{50}$$

Noether boundary conditions are

$$u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u,\ u_i \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u \underset{t\to\pm\infty}{\longrightarrow} 0\,. \tag{51}$$

The strict boundary conditions read

$$xu_t,\ xu_x^2,\ u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u_x,\ yu_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u_z \underset{z\to\pm\infty}{\longrightarrow} 0\,. \tag{52}$$

The boundary conditions for this case, (51) – (52) are weaker than in the general case (41). The following conservation law is associated with the symmetry X_α (39):

$$D_t \iint \left[t^2 u_x u_y + t y u_x^2 + x y u_x - y u\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \doteq 0\,. \tag{53}$$

$$X_h = -2h\,\frac{\partial}{\partial z} - h' z\,\frac{\partial}{\partial x} + \left[x z h'' + \tfrac{1}{4}\,h'''\left(\tfrac{1}{3}\,z^3 + z y^2\right)\right]\frac{\partial}{\partial u}\,, \tag{54}$$

$h = h(t)$ is arbitrary function. We get the following conservation laws:

$$D_t \iint u_x u_z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \doteq 0 \tag{55}$$

(conservation of z–component of the momentum of the system P_z) with the boundary conditions

$$u_i \underset{z\to\pm\infty}{\longrightarrow} 0\,, \quad u_t u_z,\, u_x^2 u_z \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u_y u_z \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad i = 1,\ldots,4\,, \tag{56}$$

$$D_t \iint \left[2 t u_x u_z + z u_x^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \doteq 0 \tag{57}$$

with the boundary conditions

$$u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u_z,\, z u_x u_z \underset{z\to\pm\infty}{\longrightarrow} 0\,, \quad i = 1,\ldots,4\,, \tag{58}$$

and

$$D_t \iint \left[t^2 u_x u_z + t z u_x^2 + x z u_x - z u\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \doteq 0 \tag{59}$$

with the boundary conditions

$$x u_t,\, x u_x^2,\, u_i \underset{x\to\pm\infty}{\longrightarrow} 0\,, \quad u_y \underset{y\to\pm\infty}{\longrightarrow} 0\,, \quad u,\, u_i,\, z u_z \underset{z\to\pm\infty}{\longrightarrow} 0\,, \quad u \underset{t\to\pm\infty}{\longrightarrow} 0\,. \tag{60}$$

We have generated a system of local conservation laws for the two–dimensional potential Zabolotskaya–Khokhlov equation (1), associated with its infinite Lie point symmetries. Infinite symmetries with arbitrary functions of time lead to a finite number of local conservation laws, and the form of each conservation law is determined by a specific form of boundary condition, as expected according to the general approach [13]. Obtained conservation laws have characteristics α as in the case of the First Noether Theorem. For the potential Zabolotskaya–Khokhlov equation we have shown that known conservation laws of momentum (33), (45), (55) correspond to the weakest boundary conditions, with vanishing function and its derivatives on the boundary (infinity). Other local conservation laws have stricter asymptotic behavior.

References

[1] E.A. Zabolotskaya and R.V. Khokhlov: Sov. Phys. Acoust. **15** (1969) 35.

[2] A.M. Vinogradov, I.S. Krasilshchik and V.V.Lychagin: in *Introduction to Geometry of Nonlinear Differential Equations*, Nauka, Moscow, 1986.

[3] F. Schwarz: J. Phys. A. **20** (1987) 1613.

[4] N.O. Sharomet: Acta Appl. Math. **15** (1989) 83.

[5] S.V. Korsunski: Mat. Fiz. Nelin. Mekh. **16** (1991) 81 (in Russian).

[6] J.-F. Zhang, Y-J. Zhu and J. Lin: Comm. Theor. Phys. **24** (1995) 69.

[7] D. David, D. Levi and P. Winternitz: Phys. Lett A. **129** (1988) 161.

[8] P. Kucharczyk: Bull. Polon. Acad. Sci., Ser. Techn. Sci. **XIII** (1965) 469.

[9] S.V. Sukhinin: Dynamika Sploshnoi Sredy **3** (1978) 130 (in Russian).

[10] N.H. Ibragimov (ed): in *Lie Group Analysis of Differential Equations*, CRC Press, Boca Raton, 1994.

[11] R.S. Khamitova: Dokl. Acad. Nauk S.S.S.R. **248** (1979) 798 (in Russian).

[12] V. Rosenhaus: in *Proc. Symmetry and Perturbation Theory 2001, Cala Gonone, Italy*, World Scientific, 2001, p.183.

[13] V. Rosenhaus: J. Math. Phys. **43** (2002) 6129.

[14] P.J. Olver: in *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.

[15] E. Noether: Nachr. König. Gessell. Wissen. Göttingen, Math.-Phys. **Kl** (1918) 235.