A division algebra classification of generalized supersymmetries

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Generalized supersymmetries admitting bosonic tensorial central charges are classified in accordance with their division algebra properties. Division algebra consistent constraints lead (in the complex and quaternionic cases) to the classes of hermitian and holomorphic generalized supersymmetries. Applications to the analytic continuation of the M–algebra to the Euclidean and the systematic investigation of certain classes of models in generic space–times are briefly mentioned.

PACS: 11.30.Pb
Key words: supersymmetry, M–theory

1 Introduction

This talk is based on a series of papers, [1, 2, 3], devoted to the investigation of generalized supersymmetries in connection with division algebras, as well as the application of these results in the broad context of the M–theory. In the last work [3], in particular, the notion of hermitian and holomorphic division–algebra constrained generalized supersymmetries for complex and quaternionic spinors was thoroughly investigated. It is not a mere mathematical curiosity prompting us in the search of a classification scheme for this type of construction, but concrete physical motivations based on M–theory and related topics. It was indeed proven in [2] that the notion of holomorphic complex generalized supersymmetry is required in order to perform the analytic continuation of the Minkowskian M–algebra to the Euclidean. Moreover, it is clear that the present results can be applied to the classification of various classes of supersymmetric dynamical systems presenting bosonic tensorial central charges (more on that later).

It is worth recalling that the problem of classifying supersymmetries has recently regained interest and found a lot of attention in the literature. We can cite, e.g., a series of papers where the notion of "spin algebra" has been introduced and investigated [4]. An even more updated reference concerns the classification of the so-defined "polyvector super–Poincaré algebras" [5].

The reasons behind all this activity are clear. In the seventies the HŁS scheme [6] was a cornerstone providing the supersymmetric extension of the Coleman–Mandula no–go theorem. However, in the nineties, the generalized space–time supersymmetries going beyond the HŁS scheme (and admitting, in particular, a bosonic sector of the Poincaré or conformal superalgebra which could no longer be expressed as a tensor product $B_{\text{geom}} \oplus B_{\text{int}}$, where B_{geom} describes space–time Poincaré or conformal algebras, while the remaining generators spanning B_{int} are scalars) found widespread recognition [7, 8] in association with the dynamics of

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extended objects like branes (see [9, 10]). The eleven–dimensional M–algebra underlying the M–theory as a possible "Theory Of Everything" (TOE), admitting 32–real component spinors and maximal number ($= 528$) of saturated bosonic generators [7, 8] falls into this class of generalized supersymmetries. The physical motivations for dealing with and classifying generalized supersymmetries are therefore quite obvious. The purely mathematical side as well presents very attracting features. The ingredients that have to be used have been known by mathematicians since at least the sixties ([11], see also [12] and, for quite a convenient presentation for physicists, [13]). They include the division–algebra classification of Clifford algebras and fundamental spinors. It is quite rewarding that, by using these available tools, we can conveniently formulate and solve the problem of classifying generalized supersymmetries.

It is well-known that the Clifford algebra irreps [13] are put in correspondence with the **R**, **C**, **H** division algebras. An analogous scheme works for fundamental spinors (here and in the following, fundamental spinors are defined to be the spinors admitting, in a given space–time, the maximal division algebra structure compatible with the minimal number of *real* components). Both the eleven-dimensional M–algebra and the F–algebra in $(10 + 2)$ dimensions are based on real spinors. Their analytic continuation to the Euclidean, however, see [2] and [3], are based on complex spinors. The presence of both complex and quaternionic spinors allows introducing division–algebra compatible extra–constraint on the available generalized supersymmetries. The reason for that lies in the fact that in these two extra cases one has at disposal the division–algebra principal conjugation (which simply coincides, for real numbers, with the identity operator) to further play with. As a consequence, the two big classes of (complex or quaternionic) constrained hermitian versus holomorphic generalized supersymmetries can be consistently introduced.

It is of particular importance to determine the biggest ("saturated") generalized supersymmetry compatible with the given division–algebra structure and constraint. The complete classification is here presented in a series of tables.

For the sake of simplicity, in this work we are only concerned with "generalized supertranslations". This means in particular that the bosonic generators are all abelian. The construction of, e.g., Lorentz generators requires a bigger algebra than the ones here examined. One viable scheme to produce them consists in introducing a generalized superconformal algebra (which, in its turn, allows recovering a generalized super–Poincaré algebra through an Inonü–Wigner type of contraction). Following [14], this can be easily achieved by taking two separate copies of "generalized supertranslations" and imposing the Jacobi identities on the whole set of generators to fully determine the associated superconformal algebra.

2 Clifford algebras and spinors

We recall here the basic features of the classification of Clifford algebras and spinors which will be useful later on.

This preliminary material about the classification of the Clifford algebras associated to the **R**, **C**, **H** associative division algebras is based on [13] and [1].

The most general irreducible *real* matrix representations of the Clifford algebra

$$\Gamma^{\mu}\Gamma^{\nu}+\Gamma^{\nu}\Gamma^{\mu}=2\eta^{\mu\nu}, \qquad (1)$$

with $\eta^{\mu\nu}$ being a diagonal matrix of (p,q) signature (i.e. p positive, $+1$, and q negative, -1, diagonal entries)[1]) can be classified according to the property of the most general S matrix commuting with all the Γ's ($[S,\Gamma^{\mu}]=0$ for all μ). If the most general S is a multiple of the identity, we get the normal (**R**) case. Otherwise, S can be the sum of two matrices, the second one multiple of the square root of -1 (this is the almost complex, **C** case) or the linear combination of 4 matrices closing the quaternionic algebra (this is the **H** case). According to [13] the *real* irreducible representations are of **R**, **C**, **H** type, according to the following table, whose entries represent the values $(p-q)\bmod 8$

R	**C**	**H**
0, 2		4, 6
1	3, 7	5

(2)

The real irreducible representation is always unique unless $(p-q)\bmod 8=1,\,5$. In these signatures two inequivalent real representations are present, the second one recovered by flipping the sign of all Γ's ($\Gamma^{\mu}\mapsto-\Gamma^{\mu}$).

Let us denote as $C(p,q)$ the Clifford irreps corresponding to the (p,q) signatures. The normal (**R**), almost complex (**C**) and quaternionic (**H**) type of the corresponding Clifford irreps can also be understood as follows. While in the **R**–case the matrices realizing the irrep have necessarily real entries, in the **C**–case matrices with complex entries can be used, while in the **H**–case the matrices can be realized with quaternionic entries.

It is worth noticing that in the given signatures $(p-q)\bmod 8=0,4,6,7$, without loss of generality, the Γ^{μ} matrices can be chosen block–antidiagonal (generalized Weyl–type matrices), i.e. of the form

$$\Gamma^{\mu}=\left(\begin{array}{cc}0 & \sigma^{\mu}\\ \tilde{\sigma}^{\mu} & 0\end{array}\right). \qquad (3)$$

In these signatures it is therefore possible to introduce the Weyl–projected spinors, whose number of components is half of the size of the corresponding Γ–matrices[2]).

A very convenient presentation of the irreducible representations of Clifford algebras makes use of an algorithm allowing to single out, in each arbitrary signature space–time, a representative (up to, at most, the sign flipping $\Gamma^{\mu}\leftrightarrow-\Gamma^{\mu}$) in each irreducible class of representations of Clifford's gamma matrices has been given in [1].

[1]) Throughout this paper it will be understood that the positive eigenvalues are associated with space-like directions, the negative ones with time–like directions.

[2]) This notion of Weyl spinors, which is convenient for our purposes, is different from the one usually adopted in connection with *complex*–valued Clifford algebras and has been introduced in [1].

At first one proves that starting from a given D spacetime–dimensional representation of Clifford's Gamma matrices, one can recursively construct $D+2$ spacetime dimensional Clifford Gamma matrices with the help of two recursive algorithms. Indeed, it is a simple exercise to verify that if γ_i's denotes the d–dimensional Gamma matrices of a $D = p + q$ spacetime with (p, q) signature (namely, providing a representation for the $C(p, q)$ Clifford algebra) then $2d$–dimensional $D + 2$ Gamma matrices (denoted as Γ_j) of a $D + 2$ spacetime are produced according to either

$$\begin{aligned} \Gamma_j \;\equiv\; & \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}, \\ (p,q) \mapsto\; & (p+1, q+1) \end{aligned} \tag{4}$$

or

$$\begin{aligned} \Gamma_j \;\equiv\; & \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}, \\ (p,q) \mapsto\; & (q+2, p)\,. \end{aligned} \tag{5}$$

As an example, one can realize that three 2×2 matrices τ_A, τ_1, τ_2 realizing the Clifford algebra $C(2, 1)$ are obtained by applying either (4) or (5) to the number 1, i.e. the one–dimensional realization of $C(1, 0)$.

The above construction can be applied to produce all irreps of Clifford algebras, by knowing some fundamental representations associated with division algebras, for details see [3]. For that reason it is convenient to review here the basic features of division algebras which will be needed in the following.

The four division algebra of real (**R**) and complex (**C**) numbers, quaternions (**H**) and octonions (**O**) possess respectively 0, 1, 3 and 7 imaginary elements e_i satisfying the relations

$$e_i \cdot e_j = -\delta_{ij} + C_{ijk} e_k \,, \tag{6}$$

(i, j, k are restricted to take the value 1 in the complex case, 1, 2, 3 in the quaternionic case and 1, 2, ..., 7 in the octonionic case; furthermore, the sum over repeated indices is understood).

C_{ijk} are the totally antisymmetric division–algebra structure constants. The octonionic division algebra is the maximal, since quaternions, complex and real numbers can be obtained as its restriction. The totally antisymmetric octonionic structure constants can be expressed as

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{7}$$

(and vanishing otherwise).

The octonions are the only non-associative, however alternative (see [15]), division algebra.

Due to the antisymmetry of C_{ijk}, it is clear that we can realize (1) by associating the $(0, 3)$ and $(0, 7)$ signatures to, respectively, the imaginary quaternions and the imaginary octonions.

For our later purposes it is of particular importance the notion of division–algebra principal conjugation. Any element X in the given division algebra can be expressed through the sum

$$X = x_0 + x_i e_i\,, \tag{8}$$

where x_0 and x_i are real, the summation over repeated indices is understood and the positive integral i are restricted up to 1, 3 and 7 in the **C**, **H** and **O** cases respectively. The principal conjugate X^* of X is defined to be

$$X^* = x_0 - x_i e_i\,. \tag{9}$$

It allows introducing the division–algebra norm through the product X^*X. The normed–one restrictions $X^*X = 1$ select the three parallelizable spheres S^1, S^3 and S^7 in association with **C**, **H** and **O** respectively.

Further comments on the division algebras and their relations with Clifford algebras can be found in [1] and [15].

The fundamental spinors carry a representation of the generalized Lorentz group with a minimal number of real components in association with the maximal, compatible, allowed division–algebra structure.

The following table, taken from the results in [4] and [13], see also [1], presents the comparison between division–algebra properties of Clifford irreps ($\mathbf{\Gamma}$) and fundamental spinors ($\mathbf{\Psi}$), in different space–times parametrized by $\rho = (s-t) \bmod 8$. We have

ρ	$\mathbf{\Gamma}$	$\mathbf{\Psi}$
0	**R**	**R**
1	**R**	**R**
2	**R**	**C**
3	**C**	**H**
4	**H**	**H**
5	**H**	**H**
6	**H**	**C**
7	**C**	**R**

(10)

It is clear from the above table that, for $\rho = 2, 3$, the fundamental spinors can accommodate a larger division–algebra structure than the corresponding Clifford irreps. Conversely, for $\rho = 6, 7$, the Clifford irreps accommodate a larger division–algebra structure than the corresponding spinors. In several cases this mismatch of division–algebra structures plays an important role. For instance in [14] a method was introduced to construct superconformal algebras based on the minimal division algebra structure common to both Clifford irreps and fundamental spinors. This method can be straightforwardly modified to produce extended superconformal algebras based on the largest division–algebra structure. The price to be paid, in this case, would imply the introduction, for $\rho = 2, 3$, of reducible Clifford representations and, conversely, for $\rho = 6, 7$ of non-minimal spinors.

The reason behind the mismatch can be easily understood on the basis of the fact that fundamental spinors are Weyl projected if the matrices realizing the Clifford algebra generators can be taken in a block antidiagonal form.

3 Generalized supersymmetries: the M and F algebra examples

Three matrices, denoted as A, B, C, have to be introduced in association with the three conjugations (hermitian, complex and transposition) acting on Gamma matrices [16]. Since only two of the above matrices are independent we choose here, following [1], to work with A and C. A plays the role of the time–like Γ^0 matrix in the Minkowskian space–time and is used to introduce barred spinors. C, on the other hand, is the charge conjugation matrix. Up to an overall sign, in a generic (s,t) space–time, A and C are given by the products of all the time–like and, respectively, all the symmetric (or antisymmetric) Gamma–matrices[3]). The properties of A and C immediately follow from their explicit construction, see [16] and [1].

In a representation of the Clifford algebra realized by matrices with real entries, the conjugation acts as the identity, see (9). In this case the space–like gamma matrices are symmetric, while the time–like gamma matrices are antisymmmetric, so that A can be identified with the charge conjugation matrix C_A.

For our purposes the importance of A and the charge conjugation matrix C lies on the fact that, in a D–dimensional space–time ($D = s + t$) spanned by $d \times d$ Gamma matrices, they allow to construct a basis for $d \times d$ (anti)hermitian and (anti)symmetric matrices, respectively. It is indeed easily proven that, in the real and the complex cases (the quaternionic case is different), the $\begin{pmatrix} D \\ k \end{pmatrix}$ antisymmetrized products of k Gamma matrices $A\Gamma^{[\mu_1\cdots\mu_k]}$ are all hermitian or all antihermitian, depending on the value of $k \leq D$. Similarly, the antisymmetrized products $C\Gamma^{[\mu_1\cdots\mu_k]}$ are all symmetric or all antisymmetric.

For what concerns the M–algebra, the 32–component real spinors of the $(10,1)$–spacetime admit anticommutators $\{Q_a, Q_b\}$ which are 32×32 symmetric real matrices with, at most, $32 + \frac{32\times 31}{2} = 528$ components. Expanding the r.h.s. in terms of the antisymmetrized product of Gamma matrices, we get that it can be saturated by the so-called M–algebra

$$\{Q_a, Q_b\} = (A\Gamma_\mu)_{ab}\, P^\mu + \left(A\Gamma_{[\mu\nu]}\right)_{ab} Z^{[\mu\nu]} + \left(A\Gamma_{[\mu_1\cdots\mu_5]}\right)_{ab} Z^{[\mu_1\cdots\mu_5]}. \qquad (11)$$

Indeed, the $k = 1,\ 2,\ 5$ sectors of the r.h.s. furnish $11 + 55 + 462 = 528$ overall components. Besides the translations P^μ, in the r.h.s. the antisymmetric rank–2 and rank–5 abelian tensorial central charges, $Z^{[\mu\nu]}$ and $Z^{[\mu_1\cdots\mu_5]}$ respectively, appear.

[3]) Depending on the given space–time (see [16] and [1]), there are at most two charge conjugations matrices, C_S, C_A, given by the product of all symmetric and all antisymmetric gamma matrices, respectively. In special space–time signatures they collapse into a single matrix C.

The (11) saturated M–algebra admits a finite number of subalgebras which are consistent with the Lorentz properties of the Minkowskian eleven dimensions. There are 6 such subalgebras which are recovered by setting either one or two among the three sets of tensorial central charges P^{μ}, $Z^{[\mu\nu]}$, $Z^{[\mu_1\cdots\mu_5]}$ identically equal to zero (a completely degenerate subalgebra is further obtained by setting the whole r.h.s. identically equal to zero).

The fact that the fundamental spinors in a $(10,2)$–spacetime also admit 32 components is due to the existence of the Weyl projection. This implies that the saturated M–algebra admits a $(10,2)$ space–time presentation, the so-called F–algebra, in terms of $(10,2)$ Majorana–Weyl spinors $\tilde{Q}_{\tilde{a}}$, $\tilde{a}=1,2,\ldots,32$.

In the case of Weyl projected spinors the r.h.s. has to be reconstructed with the help of a projection operator which selects the upper left block in a 2×2 block decomposition. Specifically, if $\mathcal{M}$ is a matrix decomposed in 2×2 blocks as $\mathcal{M}=\begin{pmatrix}\mathcal{M}_1 & \mathcal{M}_2\\ \mathcal{M}_3 & \mathcal{M}_4\end{pmatrix}$, we can define

$$P(\mathcal{M})\equiv\mathcal{M}_1\,. \tag{12}$$

The saturated M–algebra (11) can therefore be rewritten as

$$\left\{\tilde{Q}_{\tilde{a}},\tilde{Q}_{\tilde{b}}\right\}=P\left(\tilde{A}\tilde{\Gamma}_{\tilde{\mu}\tilde{\nu}}\right)_{\tilde{a}\tilde{b}}\tilde{Z}^{[\tilde{\mu}\tilde{\nu}]}+P\left(\tilde{A}\tilde{\Gamma}_{[\tilde{\mu}_1\ldots\tilde{\mu}_6]}\right)_{\tilde{a}\tilde{b}}\tilde{Z}^{[\tilde{\mu}_1\ldots\tilde{\mu}_6]}, \tag{13}$$

where all tilde's are referred to the corresponding $(10,2)$ quantities. The matrices in the r.h.s. are symmetric in the exchange $\tilde{a}\leftrightarrow\tilde{b}$. This time the rank–2 and self-dual rank–6 antisymmetric abelian tensorial central charges, $\tilde{Z}^{[\tilde{\mu}\tilde{\nu}]}$ and respectively $\tilde{Z}^{[\tilde{\mu}_1\cdots\tilde{\mu}_6]}$, appear. Their total number of components is $66+462=528$, therefore proving the saturation of the r.h.s. The saturated equation (13) is named the F–algebra.

4 Real, complex and quaternionic generalized supersymmetries

For real n–component spinors Q_a, the most general supersymmetry algebra is represented by

$$\{Q_a,Q_b\}=\mathcal{Z}_{ab}, \tag{14}$$

where the matrix $\mathcal{Z}$ appearing in the r.h.s. is the most general $n\times n$ symmetric matrix with total number of $\frac{n(n+1)}{2}$ components. For any given space–time we can easily compute its associated decomposition of $\mathcal{Z}$ in terms of the antisymmetrized products of k–Gamma matrices, namely

$$\mathcal{Z}_{ab}=\sum_k(A\Gamma_{[\mu_1\cdots\mu_k]})_{ab}Z^{[\mu_1\cdots\mu_k]}, \tag{15}$$

where the values k entering the sum in the r.h.s. are restricted by the symmetry requirement for the $a\leftrightarrow b$ exchange and are specific for the given spacetime. The coefficients $Z^{[\mu_1\cdots\mu_k]}$ are the rank–k abelian tensorial central charges.

When the fundamental spinors are complex or quaternionic they can be organized in complex (for the **C** and **H** cases) and quaternionic (for the **H** case) multiplets, whose entries are respectively complex numbers or quaternions.

The real generalized supersymmetry algebra (14) can now be replaced by the most general complex or quaternionic supersymmetry algebras, given by the anticommutators among the fundamental spinors Q_a and their conjugate ${Q^*}_{\dot{a}}$ (where the conjugation refers to the principal conjugation in the given division algebra, see (9)). We have in this case

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab}\,, \quad \{{Q^*}_{\dot{a}}, {Q^*}_{\dot{b}}\} = {\mathcal{Z}^*}_{\dot{a}\dot{b}}\,, \tag{16}$$

together with

$$\{Q_a, {Q^*}_{\dot{b}}\} = \mathcal{W}_{a\dot{b}}\,, \tag{17}$$

where the matrix $\mathcal{Z}_{ab}$ (${\mathcal{Z}^*}_{\dot{a}\dot{b}}$ is its conjugate and does not contain new degrees of freedom) is symmetric, while $\mathcal{W}_{a\dot{b}}$ is hermitian.

The maximal number of allowed components in the r.h.s. is given, for complex fundamental spinors with n complex components, by
ia) $n(n+1)$ (real) bosonic components entering the symmetric $n\times n$ complex matrix $\mathcal{Z}_{ab}$ plus
iia) n^2 (real) bosonic components entering the hermitian $n \times n$ complex matrix $\mathcal{W}_{a\dot{b}}$.

Similarly, the maximal number of allowed components in the r.h.s. for quaternionic fundamental spinors with n quaternionic components is given by
ib) $2n(n+1)$ (real) bosonic components entering the symmetric $n \times n$ quaternionic matrix $\mathcal{Z}_{ab}$ plus
iib) $2n^2 - n$ (real) bosonic components entering the hermitian $n \times n$ quaternionic matrix $\mathcal{W}_{a\dot{b}}$.

The previous numbers do not necessarily mean that the corresponding generalized supersymmetry is indeed saturated. This is in particular true in the quaternionic case, see [3].

Any real generalized supersymmetry admitting a complex structure can be re-expressed in a complex formalism with n–component complex spinors and total number of $n(2n + 1)$ (real) bosonic components split into $n(n + 1)$ components entering the symmetric matrix $\mathcal{Z}$ and n^2 components entering the hermitian matrix $\mathcal{W}$. The situation is different in the quaternionic case. The quaternionic structure requires a restriction on the total number of bosonic generators. n–component quaternionic spinors can be described as $4n$–component real spinors. However, the r.h.s. of a quaternionic (16) and (17) superalgebra admits at most $4n^2 + n$ bosonic components, instead of $8n^2 + 2n$ of the most general supersymmetric real algebra. The Lorentz–covariance further restricts the number of bosonic generators in a quaternionic supersymmetry algebra.

We conclude this section mentioning the two big classes of subalgebras, respecting the Lorentz–covariance, that can be obtained from (16) and (17) in both the complex and quaternionic cases. They are obtained by setting identically equal to zero either $\mathcal{Z}$ or $\mathcal{W}$, namely

I) $\mathcal{Z}_{ab} \equiv \mathcal{Z}^*{}_{\dot{a}\dot{b}} \equiv 0$, so that the only bosonic degrees of freedom enter the hermitian matrix $\mathcal{W}_{a\dot{b}}$ or, conversely,

II) $\mathcal{W}_{a\dot{b}} \equiv 0$, so that the only bosonic degrees of freedom enter $\mathcal{Z}_{ab}$ and its conjugate matrix $\mathcal{Z}^*{}_{\dot{a}\dot{b}}$.

Accordingly, in the following we will refer to the (complex or quaternionic) generalized supersymmetries satisfying the *I)* constraint as "hermitian" (or "type *I*") generalized supersymmetries, while the (complex or quaternionic) generalized supersymmetries satisfying the *II)* constraint will be referred to as "holomorphic" (or "type *II*") generalized supersymmetries.

5 Some examples of consistently constrained complex generalized supersymmetries

Generalized supersymmetries can be classified according to their division–algebra character $\mathbf{Y}$ (with $\mathbf{Y} \equiv \mathbf{R}, \mathbf{C}, \mathbf{H}$). They can be conveniently labelled with a pair of division algebras as "$\mathbf{XY}$", where $\mathbf{X}$ specifies whether spinors are realized as column vectors of real numbers ($\mathbf{X} = \mathbf{R}$), complex numbers ($\mathbf{X} = \mathbf{C}$) or quaternions ($\mathbf{X} = \mathbf{H}$). Accordingly, generalized supersymmetries fall into different cases:

i) **RR**,

ii) **RC** and **CC**,

iii) **RH**, **CH** and **HH**.

In the **CC**, **CH** and **HH** cases a suffix can be added, specifying whether we are dealing with a hermitian (type I, therefore $\mathbf{CC}_I$, $\mathbf{CH}_I$, $\mathbf{HH}_I$) or a holomorphic (type II, $\mathbf{CC}_{II}$, $\mathbf{CH}_{II}$, $\mathbf{HH}_{II}$) generalized supersymmetry. A closer inspection shows that the following identities hold for hermitian supersymmetries

$$\mathbf{RC} \equiv \mathbf{CC}_I \tag{18}$$

and

$$\mathbf{RH} \equiv \mathbf{CH}_I \equiv \mathbf{HH}_I \,. \tag{19}$$

The first identity means that representing complex spinors in real notations is tantamount to realize a complex hermitian supersymmetry. The second set of identities holds for supersymmetries realized with quaternionic spinors.

In the following, for simplicity, it will be symbolically denoted as "$\mathbf{M}_k$" the space of $\begin{pmatrix} D \\ k \end{pmatrix}$–component, totally antisymmetric rank–k tensors of a D–dimensional spacetime, associated to the basis provided by the hermitian $A\Gamma^{[\mu_1\cdots\mu_k]}$ matrices (namely, entering "type I" supersymmetries). Similarly, the rank–k totally antisymmetric tensors associated to the symmetric matrices $C\Gamma^{[\mu_1\cdots\mu_k]}$ and entering the type II, holomorphic, supersymmetries will be denoted as "$\mathcal{M}_k$" (the symbol "M_k" will be reserved to real, "$\mathbf{RY}$", supersymmetries).

It is quite convenient to illustrate how complex and quaternionic supersymmetries work by discussing specific examples. The extension of both reasonings and results to general spacetimes is in fact guaranteed by the already mentioned algorithmic construction. We illustrate here the example of the supersymmetries associated to the $(4,1)$ spacetime and its dimensional reduction to the usual Minkowski $(3,1)$ case.

$(4,1)$–dimensional real spinors possess eight components and can be regarded as spinors of the extended $(4,3)$ spacetime, see [3]. It can be easily checked that in $D=7$ (for the $(4,3)$ space–time) dimensions, the bosonic sector of the supersymmetry algebra is given by the $1+35=36$ rank–k tensors ${M_0}^{(D=7)}+{M_3}^{(D=7)}$. Expanding these tensors in the $D=5$–dimensional ($(4,1)$ spacetime) basis we are led to the following identifications

$$
{M_0}^{(D=7)}+{M_3}^{(D=7)} \equiv {M_0}^{(D=5)}+{M_3}^{(D=5)}+2\times {M_2}^{(D=5)}+{M_1}^{(D=5)}\,, \qquad (20)
$$

where the counting of the components reads as follows

$$
1+35=1+10+2\times 10+5\,. \qquad (21)
$$

The equation (20) above corresponds to the saturated bosonic sector of the **RR** generalized supersymmetry in a $(4,1)$ spacetime.

Let us discuss now the two complex supersymmetries ($\mathbf{CC}_I$ and $\mathbf{CC}_{II}$) associated with the $(4,1)$ spacetime.

It can be easily shown that

i) in the $\mathbf{CC}_I$ case the bosonic sector is expressed as

$$
\mathbf{M}_1+\mathbf{M}_3+\mathbf{M}_5\,. \qquad (22)
$$

The expected 16 bosonic components (real counting) of the saturated complex hermitian algebra are indeed recovered through

$$
16=5+10+1\,; \qquad (23)
$$

it should be noticed that the rank k antisymmetric tensors *are not* related by the Hodge duality; *ii)* in the $\mathbf{CC}_{II}$ case the bosonic sector is expressed as

$$
\mathcal{M}_2+\mathcal{M}_3\,, \qquad (24)
$$

whose total number of bosonic components, $10+10=20$, indeed saturates the number of bosonic components for the complex holomorphic supersymmetry; in this case as well the rank–2 and rank–3 bosonic tensors *are not* related by Hodge duality (indeed one sector is real while the other one is completely imaginary since the product of the five distinct gamma matrices is proportional to i). However, a reality constraint can be *further imposed* on the bosonic sector of $\mathbf{CC}_{II}$. If this Lorentz–consistent constraint is applied, the total number of bosonic components corresponds to half the number of saturated bosonic components of the complex holomorphic supersymmetry. This consistent reduction is a common feature of all complex holomorphic supersymmetries and not a special case of just the $(4,1)$ spacetime.

It should be noticed that the 36 bosonic components of the saturated $(4,1)$ **RR** supersymmetry are recovered from the $16+20$ bosonic components of the saturated hermitian and holomorphic supersymmetries. In a loose notation we can symbolically write

$$\mathbf{RR} \approx \mathbf{CC}_I + \mathbf{CC}_{II}\,. \qquad (25)$$

By using complex spinors in the $(4,1)$ spacetime we end up with the following list of consistent division algebra constraints that can be imposed on the generalized complex supersymmetries. We have the following table of generalized supersymmetries, with their associated number of bosonic components (in the real counting), in a $(4,1)$ spacetime

$$\begin{array}{rcl} \text{full supersymmetry} & \equiv & 36 \text{ components}\,,\\ \text{hermitian supersymmetry} & \equiv & 16 \text{ components}\,,\\ \text{holomorphic supersymmetry} & \equiv & 20 \text{ components}\,,\\ \text{(restricted) holomorphic supersymmetry} & \equiv & 10 \text{ components}\,,\\ \text{herm. + (restr.) holom. supersymmetry} & \equiv & 26 \text{ components}\,. \end{array} \qquad (26)$$

In the above table, and similarly in the one below, the "restricted holomorphic supersymmetry" is realized by implementing a reality condition on the bosonic r.h.s. of the holomorphic supersymmetry.

An analogous table can be produced in the $(3,1)$ spacetime, for 2–component complex Weyl spinors. We can write down the following list of division–algebra constrained supersymmetries

$$\begin{array}{rcl} \text{full supersymmetry} & \equiv & 10 \text{ components}\,,\\ \text{hermitian supersymmetry} & \equiv & 4 \text{ components}\,,\\ \text{holomorphic supersymmetry} & \equiv & 6 \text{ components}\,,\\ \text{(restricted) holomorphic supersymmetry} & \equiv & 3 \text{ components}\,,\\ \text{herm. + (restr.) holom. supersymmetry} & \equiv & 7 \text{ components}\,. \end{array} \qquad (27)$$

Similar decompositions work in any other space–times supporting complex spinors. A classification of such supersymmetries can be performed also in the case of quaternionic spacetimes (supporting quaternionic spinors). The results are reported in the next section.

6 Generalized supersymmetries of the quaternionic spacetimes

We present here the classification of quaternionic generalized supersymmetries associated to quaternionic space–times carrying quaternionic fundamental spinors.

The following results do not depend on the signature of the space–time, but only on its dimensionality D. Let us start with the hermitian quaternionic supersymmetry $\mathbf{HH}_I$. In association with each one of the quaternionic spacetimes up to $D = 13$ ([3]) the bosonic sector is decomposed in rank–k antisymmetric tensors, with total number of (real counting) bosonic components according to the table

spacetime	bosonic sectors	bosonic components
$D=3$	$\mathbf{M}_0$	1
$D=4$	$\mathbf{M}_0$	1
$D=5$	$\mathbf{M}_0+\mathbf{M}_1$	$1+5=6$
$D=6$	$\mathbf{M}_1$	6
$D=7$	$\mathbf{M}_1+\mathbf{M}_2$	$7+21=28$
$D=8$	$\mathbf{M}_2$	28
$D=9$	$\mathbf{M}_2+\mathbf{M_3}$	$36+84=120$
$D=10$	$\mathbf{M}_3$	120
$D=11$	$\mathbf{M}_0+\mathbf{M}_3+\mathbf{M}_4$	$1+165+330=496$
$D=12$	$\mathbf{M}_0+\mathbf{M}_4$	$1+495=496$
$D=13$	$\mathbf{M}_0+\mathbf{M}_1+\mathbf{M}_4+\mathbf{M}_5$	$1+13+715+1287=2016$

(28)

Please notice from the above table that the hermitian quaternionic supersymmetry saturates the bosonic sector, as expected.

Let us now discuss the holomorphic supersymmetries associated with the quaternionic spacetimes. The complex holomorphic supersymmetry $\mathbf{CH}_{II}$ is characterized by the table

spacetime	bosonic sectors	bosonic components
$D=3$	$\mathcal{M}_1$	3
$D=4$	$\widetilde{\mathcal{M}}_2$	3
$D=5$	$\mathcal{M}_2$	10
$D=6$	$\widetilde{\mathcal{M}}_3$	10
$D=7$	$\mathcal{M}_0+\mathcal{M}_3$	$1+35=36$
$D=8$	$\mathcal{M}_0+\widetilde{\mathcal{M}}_4$	$1+35=36$
$D=9$	$\mathcal{M}_0+\mathcal{M}_1+\mathcal{M}_4$	$1+9+126=136$
$D=10$	$\mathcal{M}_1+\widetilde{\mathcal{M}}_5$	$10+126=136$
$D=11$	$\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_5$	$11+55+462=528$
$D=12$	$\mathcal{M}_2+\widetilde{\mathcal{M}}_6$	$66+462=528$
$D=13$	$\mathcal{M}_2+\mathcal{M}_3+\mathcal{M}_6$	$78+286+1716=2080$

(29)

The tilde on the rank–k (for $k=D/2$) sectors $\widetilde{\mathcal{M}}_{D/2}$ specifies that they are self-dual (as such, their total number of bosonic components, in the real counting, is given by $\frac{1}{2}\begin{pmatrix} D \\ D/2 \end{pmatrix}$).

It should be noticed that the total counting of bosonic components in the third column implies that the $\mathbf{CH}_{II}$ superalgebras admit [3] half the number of bosonic components expected for complex spinors of the corresponding size. The recognition of this property becomes quite important when applied to the $D=11$ and $D=12$ rows of the table above. Their total number of bosonic components ($528=\frac{1}{2}\times 1056$) coincides with the number of bosonic components entering the M–algebra (11) and the F–algebra (13).

The last table is devoted to the quaternionic holomorphic supersymmetries $\mathbf{HH}_{II}$. According to [3], we can state as a theorem that quaternionic holomorphic supersymmetries *do not* involve bosonic tensorial central charges. The only admissible sectors are given by

$-$	$D = 0, 6, 7$	mod 8
$\mathcal{M}_0$	$D = 1$	mod 8
$\mathcal{M}_1$	$D = 4, 5$	mod 8
$\mathcal{M}_0 + \mathcal{M}_1$	$D = 2, 3$	mod 8

(30)

The above results can be interpreted as follows. Quaternionic holomorphic $\mathbf{HH}_{II}$ supersymmetries only arise in D-dimensional quaternionic space–times, where $D = 2,\ 3,\ 4,\ 5 \bmod 8$. No $\mathbf{HH}_{II}$ supersymmetry exists in $D = 0,\ 6,\ 7 \bmod 8$ D–dimensional spacetimes.

In $D = 1 \bmod 8$ dimensions, $\mathbf{HH}_{II}$ supersymmetries only involve a single bosonic charge. In this respect they fall into the class of quaternionic supersymmetric quantum mechanics, rather than supersymmetric relativistic theories.

Finally, the $\mathbf{HH}_{II}$ supersymmetry algebra only admits a bosonic central charge in D–dimensional quaternionic spacetimes for $D = 2,\ 3 \bmod 8$.

7 Conclusions

This paper was devoted to perform a classification of (real, complex and quaternionic) generalized supersymmetries. The notion of hermitian (complex and quaternionic) and holomorphic (complex and quaternionic) supersymmetries, as consistently division–algebra constrained generalized supersymmetries, has been presented. These supersymmetries have been classified and their main properties have been reported in a series of tables.

Physical implications of these mathematical structures are quite obvious. The classification of generalized supersymmetries allow to understand the web of interrelated dualities of different classes of theories which can be either analytically continued (let's say, to the Euclidean) or recovered through dimensional reduction.

As an example, we can cite that the analytic continuation of the M algebra was proven in [2] to correspond to an eleven–dimensional complex holomorphic supersymmetry. It was further shown in [3] that the same algebra also admits a 12–dimensional Euclidean presentation in terms of Weyl–projected spinors. These two examples of Euclidean supersymmetries can find application in the functional integral formulation of higher–dimensional supersymmetric models.

There is an interesting class of models which nicely fits in the framework here described and is currently under intense investigation. It is the class of superparticle models, introduced at first in [17] and later studied in [18], whose bosonic coordinates correspond to tensorial central charges. It was shown in [19] that a 4–dimensional theory of this kind leads to a tower of massless higher spin states, concretely implementing a Fronsdal's proposal [20] of introducing bosonic tensorial coordinates to describe massless higher spin theories (admitting helicity states greater than two). This is an active area of investigation, the main motivation beingthe investigation the tensionless limit of superstring theory, corresponding to a tower of higher helicity massless particles (see e.g. [21]).

In a somehow "orthogonal" direction, a class of theories which can be investigated in the present framework is the class of supersymmetric extensions of Chern–Simon supergravities in higher dimensions, requiring as a basic ingredient a Lie superalgebra admitting a Casimir of appropriate order, see e.g. [22].

Acknowledgement. I have profited of precious discussions with H.L. Carrion, J. Lukierski, M. Rojas and of helpful remarks by Z. Kuznetsova.

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