Note on path integration in a space with a dispiration

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Path integration is carried out in the field of topological defects. The topological defects being considered include a screw dislocation and a disclination in solid. The screw dislocation give rise to torsion, while the disclination generates curvature in the surrounding space. We consider a particle bound in the vicinity of the defect by a short range repulsive and long range attractive force. By path integration we obtain the energy spectrum and the corresponding eigenfunctions.

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1 Introduction

We consider a simple topological defect in three dimensions whose geometry is characterized by the spatial line element [6–8]

$$\mathrm{d}l^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + \sigma^2 r^2 \mathrm{d}\theta^2 + (\mathrm{d}z + \beta\mathrm{d}\theta)^2 \,, \tag{1}$$

where (r, θ, z) are cylindrical coordinates and σ and β are parameters. This describes an infinitely long linear dispiration oriented along the z-axis, with two conical singularities at the origin as seen in (3) and (4). The three dimensional geometry of the medium is characterized by nontrivial torsion and curvature which are identified with the surface density of the Burgers and Frank vectors, respectively. The Burgers vector can be viewed as a flux of torsion and the Frank vector as a flux of curvature. Since the line element of the metric is known, we can deduce the dual basis vectors

$$e^1 = \mathrm{d}z + \beta\mathrm{d}\theta\,, \quad e^2 = \mathrm{d}r \ \text{ and } \ e^3 = \sigma r \mathrm{d}\theta\,. \tag{2}$$

Observe that the periodic boundary condition in the space of (1) around the z-axis has periodicity of $2\pi\sigma$instead of 2π. This corresponds to the removal ($\sigma < 1$) or insertion ($\sigma > 1$) of a wedge of material of angle $\lambda = 2\pi(\sigma - 1)$. The Ricci scalar is given by [1]

$$R^{12}{}_{12} = R^2{}_2 = 2\pi\,\frac{1-\sigma}{\sigma}\,\delta(\rho)\,, \tag{3}$$

where $\sigma = (1 + \lambda/2\pi)$. From (3) it follows that if $0 < \sigma < 1$ the defect carries positive curvature and if $1 < \sigma < \infty$ the defect carries negative curvature.

The only non-vanishing component of torsion is given by the 2-form [3]

$$T^1 = 2\pi\beta\delta(\rho)\mathrm{d}\rho \wedge \mathrm{d}\theta\,. \tag{4}$$

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The Burgers vector is found by integrating around a closed path C encircling the dislocation [4]

$$b^1 = \oint_C h^1 = \int_S T^a = 2\pi\beta \tag{5}$$

implying that $2\pi\beta$ is the flux intensity of the torsion source passing through a closed loop C. This torsion source causes a topological change in the space where the particle propagates.

2 Lagrangian

The Lagrangian for an electron moving around the dispiration in a scalar potential $V(r)$ and a vector potential $\mathbf{A}(\overrightarrow{r})$ is

$$L = \frac{1}{2}\,M\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{1}{2}\,M\left\{\dot{r}^2 + \sigma^2 r^2\dot{\theta}^2 + \left(\dot{z} + \beta\dot{\theta}\right)^2\right\} - V(r) + e\dot{\mathbf{r}}\cdot\mathbf{A}\,. \tag{6}$$

Suppose that the dispiration and the constant magnetic Φ flux are confined in a thin impenetrable tube of radius a. For simplicity, however, we assume that the tube is so thin it can be regarded as a singular line coinciding with the dispiration line ($a \to 0$). In order to reflect the impenetrable nature of the tube, we shall include a short range repulsive force. The vector potential outside the flux tube is given by

$$\mathbf{A} = \frac{\Phi}{2\pi\sigma r}\,\widehat{e}_\theta\,. \tag{7}$$

With the aid of the above equation, the vector potential term in the Lagrangian (5) can be written as $e\dot{\mathbf{r}}\cdot\mathbf{A} = \alpha\dot{\chi}$. In this case, the Lagrangian is written as

$$L = \frac{1}{2}\,M\left\{\dot{\mathbf{r}}^2 + (\dot{z} + \beta'\dot{\chi})^2\right\} + \alpha\hbar\dot{\chi} - V(r)\,, \tag{8}$$

where $\chi = \sigma\theta$, $\beta' = \dfrac{\beta}{\sigma}$ and we made use of the two-dimensional squared velocity

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\chi}^2 \tag{9}$$

and $\alpha = \dfrac{q\Phi}{2\pi\sigma\hbar c} = \phi/\sigma$, where ϕ the magnetic flux.

3 Propagator

The transition amplitude (propagator) for the three dimensional motion of the particle from point $\mathbf{x}' = (r', \chi', z')$ to point $\mathbf{x}'' = (r'', \chi'', z'')$ can be calculated by the path integral

$$K\left(\mathbf{x}'', \mathbf{x}'; \tau\right) = \int_{x'=x(t')}^{x''=x(t'')} \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} L\,\mathrm{d}t\right\}\mathcal{D}^3\mathbf{x} \tag{10}$$

where $\tau = t'' - t'$. The path integral is explicitly expressed as

$$K\left(r'', z''; r', z'; \tau\right) = \int_{r'=r(t')}^{r''=r(t'')} \int_{z'=z(t')}^{z''=z(t'')} \mathcal{D}z\mathcal{D}^2 r \times \\ \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left\{\frac{M}{2}\,\dot{\mathbf{r}}^2 + \alpha\hbar\dot{\chi} - V(r) + \frac{M}{2}\left(\dot{z} + \beta'\dot{\chi}\right)^2\right\}\mathrm{d}t\right]. \tag{11}$$

We first calculate the z-integration by letting $\zeta = z + \beta'\chi$. Carrying out the integration yields

$$\int_{\zeta'=\zeta(t')}^{\zeta''=\zeta(t'')} \exp\left[\frac{\mathrm{i}M}{2\hbar}\int_{t'}^{t''}\dot{\zeta}^2\mathrm{d}t\right]\mathcal{D}\zeta = \\ = \lim N \to \infty \int \prod_{j=1}^{N} \exp\left[\frac{\mathrm{i}M}{2\hbar}\,\frac{\left(\zeta_j - \zeta_{j-1}\right)^2}{\epsilon}\right] \prod_{j=1}^{N-1} \mathrm{d}\zeta_j = \\ = \sqrt{\frac{\mathrm{i}M}{2\pi\hbar\tau}} \exp\left[\frac{\mathrm{i}M\left(\zeta'' - \zeta'\right)^2}{2\pi\tau}\right], \tag{12}$$

where $\tau = t'' - t'$. We may re-express the R.H.S. of (12) as

$$\sqrt{\frac{\mathrm{i}M}{2\pi\hbar\tau}} \exp\left[\frac{\mathrm{i}M\left(\zeta'' - \zeta'\right)^2}{2\pi\tau}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\hbar\tau k^2/2M}\mathrm{e}^{\mathrm{i}\left(\zeta''-\zeta'\right)k}\mathrm{d}k\,, \tag{13}$$

where $\hbar k$ is the z component of momentum of the particle. We also notice that

$$\zeta'' - \zeta' = z'' - z' + \beta'\left(\chi'' - \chi'\right) = z'' - z' + \beta'\int_{t'}^{t''}\dot{\chi}\mathrm{d}t\,. \tag{14}$$

Incorporating these results into the path integral (10), we decompose the propagator as

$$K\left(\mathbf{x}'', \mathbf{x}'; \tau\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \mathrm{e}^{\mathrm{i}k\left(z''-z'\right)}\mathrm{e}^{-\mathrm{i}\hbar\tau k^2/2M} K^{(k)}\left(\mathbf{r}'', \mathbf{r}'; \tau\right)\,, \tag{15}$$

where the two-dimensional propagator for a fixed k-value is

$$K^{(k)}\left(\mathbf{r}'', \mathbf{r}'; \tau\right) = \int_{r'=r(t')}^{r''=r(t'')} \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\,\dot{\mathbf{r}}^2 + \left(\alpha + \beta' k\right)\hbar\dot{\chi} - V(r)\right]\mathrm{d}t\right\}\mathcal{D}^2\mathbf{r}\,. \tag{16}$$

4 Path integration in the covering space

Usually the angular variable χ varies from 0 to $2\pi\sigma$. Hence, $0 \le \Delta\chi \le 2\pi\sigma$ for any angular difference. As was mentioned earlier however, the line dispiration which is described as a line singularity at $r = 0$ in (2) makes the topology of the surrounding space non-simply connected. The particle can loop around the singular line many times, requiring $\Delta\chi$ to vary from $-\infty$ to ∞. Therefore, angular path integration over the usual range gives a partial propagator due to a class of paths which do not turn around the dispiration line. For the full propagator, we have to include the contributions from paths belonging to all homotopically different classes. To carry out the angular path integration, we may either stay in the physical space M by assuming [5]

$$\int_{t'}^{t''} \dot{\chi}\mathrm{d}t = \chi'' - \chi' + 2n\pi\sigma \tag{17}$$

with $0 \leq \chi < 2\pi\sigma$, or equivalently, go over to the covering space M^* by assuming [11]

$$\int_{t'}^{t''} \dot{\chi}\mathrm{d}t = \chi'' - \chi' \tag{18}$$

with $-\infty < \chi < \infty$. In (17) $n \in \mathbf{Z}$ is the winding number.

Momentarily ignoring the winding number, we consider the path integral over the covering space. The partial propagator belonging to a class of homotopically equivalent paths which turn around the dispiration by an angle φ, $-\infty < \varphi < \infty$, may be obtained by subjecting the path integral (15) to the constraint

$$\int_{t'}^{t''} \dot{\chi}\mathrm{d}t = \varphi\,. \tag{19}$$

Specifically,

$$K_{\varphi}^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau) = \\ = \int_{r'=r(t')}^{r''=r(t'')} \delta\left(\varphi - \int_{t'}^{t''} \dot{\chi}\mathrm{d}t\right) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\,\dot{\mathbf{r}}^2 + (\alpha + \beta' k)\hbar\dot{\chi} - V(r)\right]\mathrm{d}t\right\}\mathcal{D}^2\mathbf{r}\,. \tag{20}$$

By letting

$$\delta\left(\varphi - \int_{t'}^{t''} \dot{\chi}\mathrm{d}t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{\mathrm{i}\varphi\lambda - \int_{t'}^{t''} \lambda\dot{\chi}\mathrm{d}t\right\}\mathrm{d}\lambda \tag{21}$$

we can write (20) as

$$\begin{aligned} K_{\varphi}^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{r'=r(t')}^{r''=r(t'')} \mathrm{e}^{\mathrm{i}\varphi\lambda}\times \\ \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\,\dot{\mathbf{r}}^2 - (\lambda - \alpha\hbar - \beta' k)\,\dot{\chi} - V(r)\right]\mathrm{d}t\right\}\mathcal{D}^2\mathbf{r}\mathrm{d}\lambda\,. \end{aligned} \tag{22}$$

It is clear that integration of this constrained propagator over all possible values of φ results in the full propagator (with a fixed value of k),

$$K^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau) = \int_{-\infty}^{\infty} K_{\varphi}^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau)\,\mathrm{d}\varphi\,, \tag{23}$$

which takes into account the contributions from all homotopically possible paths.

5 Angular integration

To carry out the angular path integration in (21), we express the constrained propagator in discretized form

$$K_{\varphi}^{(k)}\left(\mathbf{r}'',\mathbf{r}';\tau\right)=\frac{1}{2\pi}\lim_{N\to\infty}\int_{-\infty}^{\infty}\mathrm{d}\lambda\int_{r'=r(t')}^{r''=r(t'')}\mathrm{e}^{\mathrm{i}\varphi\lambda}\prod_{j=1}^{N}K^{(k)}(\mathbf{r}_j,\mathbf{r}_{j-1};\epsilon)\prod_{j=1}^{N-1}\mathcal{D}^2\mathbf{r}_j \tag{24}$$

with the propagator for a short time interval $\epsilon=t_j-t_{j-1}=\tau/N$,

$$K^{(k)}(\mathbf{r}_j,\mathbf{r}_{j-1};\epsilon)=\frac{M}{2\pi\mathrm{i}\hbar\epsilon}\exp\left(\frac{\mathrm{i}}{\hbar}S_j\right), \tag{25}$$

where

$$S_j=\int_{t_{j-1}}^{t_j}\left[\frac{M}{2}\dot{\mathbf{r}}^2-(\lambda-\alpha\hbar-\beta'k)\,\dot{\chi}-V(r)\right]\mathrm{d}t\,. \tag{26}$$

The pre-exponential factor in (25) is a normalization constant determined by the condition

$$\lim_{\epsilon\to 0}K^{(k)}\left(\mathbf{r}_j,\mathbf{r}_{j-1};\epsilon\right)=\delta\left(\mathbf{r}_j-\mathbf{r}_{j-1}\right). \tag{27}$$

The first term in the short time action (26) may be written as

$$\int_{t_{j-1}}^{t_j}\frac{M}{2}\dot{\mathbf{r}}^2\mathrm{d}t=\frac{M}{2\epsilon}\left(\mathbf{r}_j-\mathbf{r}_{j-1}\right)^2=\frac{M}{2\epsilon}\left[r_j^2+r_{j-1}^2-2r_jr_{j-1}\cos\left(\Delta\chi_j\right)\right]. \tag{28}$$

Since for the short time action it is sufficient to consider the contributions up to first order in ϵ, we utilize the following approximate relation valid for small ϵ, [5],

$$\cos(\Delta\chi)\simeq\cos(\Delta\chi+a\epsilon)+a\epsilon\Delta\chi+\tfrac{1}{2}\,a^2\epsilon^2 \tag{29}$$

to write the short time action as

$$S_j=\frac{M}{2\epsilon}\left(r_j^2+r_{j-1}^2\right)-\frac{M}{\epsilon}\,r_jr_{j-1}\cos\left(\Delta\chi_j-\frac{\lambda'\hbar\epsilon}{Mr_jr_{j-1}}\right)-\frac{\lambda'^2\hbar^2\epsilon^2}{4Mr_jr_{j-1}}-\epsilon V(r_j)\,, \tag{30}$$

where $\lambda'=\lambda-\alpha-\beta'k$. Combining the Jacobi–Anger expansion formula

$$\exp(z\cos\chi)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\chi}I_m(z) \tag{31}$$

and the Edward–Gluyaev asymptotic formula for the modified Bessel function

$$I_m(z)\simeq\frac{1}{\sqrt{2\pi z}}\exp\left[z-\frac{1}{2z}\left(m^2-\frac{1}{4}\right)\right] \tag{32}$$

valid for large z and for $\arg|(z)|<\pi/2$, we can derive the following asymptotic relation [2]

$$\exp\left\{z\cos\left[\chi+\mathrm{i}\frac{\lambda}{z}\right]-\frac{\lambda^2}{2z}\right\}\simeq\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\chi}I_{m+\lambda}(z)\,. \tag{33}$$

Observe that to express (31) in θ we take $m = l/\sigma$ so that $m\chi = l\theta$ and we perform the sum over l. Making use of (33) we find

$$\exp\left(\frac{\mathrm{i}}{\hbar} S_j\right) = \sum_{m_j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m_j(\chi_j - \chi_{j-1})} R_{m_j+\lambda'}\left(r_j, r_{j-1}; \epsilon\right), \tag{34}$$

where

$$R_{m_j+\lambda'}\left(r_j, r_{j-1}; \epsilon\right) = \exp\left[\frac{\mathrm{i}M}{2\hbar\epsilon}\left(r_j^2 + r_{j-1}^2\right) - \frac{\mathrm{i}\epsilon}{\hbar} V(r_j)\right] I_{|m_j+\lambda'|}\left(\frac{M r_j r_{j-1}}{\mathrm{i}\hbar\epsilon}\right). \tag{35}$$

Now the angular integration can be carried out, reducing (23) into the form

$$K^{(k)}\left(r'', \chi''; r', \chi'; \tau\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp\Big[\mathrm{i}m\left(\chi'' - \chi'\right) + \mathrm{i}\lambda\varphi\Big] Q_{m+\lambda-\alpha-\beta' k}\mathrm{d}\lambda\,, \tag{36}$$

where

$$Q_\nu = \lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\hbar\epsilon}\right)^{N-1} \int \prod_{j=1}^{N} R_\nu(r_j, r_{j-1}; \epsilon) \prod_{j=1}^{N-1} r_j \mathrm{d}r_j\,. \tag{37}$$

The full propagator with a fixed wave number k can be obtained by integrating (36) over the covering space variable φ:

$$K^{(k)}(r'', \chi''; r', \chi'; \tau) = \int_{-\infty}^{\infty} K_\varphi^{(k)}(r'', \chi''; r', \chi'; \tau)\,\mathrm{d}\varphi = \sum_{m=-\infty}^{\infty} K_m(r'', \chi''; r', \chi'; \tau) \tag{38}$$

with the partial propagator

$$K_m(r'', \chi''; r', \chi'; \tau) = \mathrm{e}^{\mathrm{i}m(\chi'' - \chi')}\, Q_{m-\alpha-\beta' k}(r'', r'; \tau)\,. \tag{39}$$

Equation (38) gives the partial wave expansion in two dimensions, and the partial propagator (39) corresponds to the propagator for the m-th partial wave.

6 The winding number representation

By letting $\lambda \to \lambda - m + \alpha + \beta' k$ the propagator (35) for a fixed angle φ may also be written as

$$K_\varphi^{(k)}(r'', \chi''; r', \chi'; \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\chi''-\chi'-\varphi)+\mathrm{i}(\lambda+\alpha+\beta' k)\varphi} Q_\lambda(r'', r'; \tau)\mathrm{d}\lambda\,. \tag{40}$$

By means of Poison's sum formula,

$$\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\chi} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\chi + 2\pi\sigma n)\,, \tag{41}$$

the above propagator can be cast into the form

$$K^{(k)}_{\varphi}(r'',\chi'';r',\chi';\tau)=\sum_{n=-\infty}^{\infty}\delta(\chi''-\chi'-\varphi+2\pi\sigma n)\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda\varphi}Q_{\lambda}(r'',r';\tau)\mathrm{d}\lambda\,. \tag{42}$$

Again, integration of (42) over φ yields

$$K^{(k)}(r'',\chi'';r',\chi';\tau)=\sum_{n=-\infty}^{\infty}K_n(r'',\chi'';r',\chi';\tau) \tag{43}$$

with

$$K_n(r'',\chi'';r',\chi';\tau)=\mathrm{e}^{\mathrm{i}(\alpha+\beta'k)(\chi''-\chi'+2\pi\sigma n)}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda(\chi''-\chi'+2\pi\sigma n)}Q_{\lambda}(r'',r';\tau)\,\mathrm{d}\lambda\,. \tag{44}$$

This is the winding number expansion [10] of the full propagator with a fixed k. The partial propagator (44) with a fixed winding number n is the propagator for the particle moving around the dislocation line n times.

7 Radial path integration

To carry out the radial integration, it is necessary to specify the shape of the potential $V(r)$. We assume that the particle in the field of the screw dislocation is subjected to a short range repulsion due to the core effect and bound by a long range attraction. As an exactly solvable example [5], we consider the spherical potential,

$$V(r)=\frac{1}{2}\,M\omega^2r^2+\frac{b\hbar^2}{2Mr^2}\,. \tag{45}$$

The short time radial propagator (35) now reads

$$\begin{aligned}R_{\nu}(r_j,r_{j-1};\epsilon)=\exp\Bigg[&\frac{\mathrm{i}M}{2\hbar\epsilon}\,(r_j^2+r_{j-1}^2)-\\&-\frac{\mathrm{i}\epsilon}{\hbar}\left(\frac{M\omega^2}{4}\,\left(r_j^2+r_{j-1}^2\right)+\frac{b}{r_jr_{j-1}}\right)\Bigg]I_{\nu}\left(\frac{Mr_jr_{j-1}}{\mathrm{i}\hbar\epsilon}\right),\end{aligned} \tag{46}$$

where $\nu=|m+\lambda-\alpha-\beta'k|$. With the help of the asymptotic recombination formula

$$I_{\nu}(az)\mathrm{e}^{c'/z}\Leftrightarrow\frac{1}{\sqrt{2\pi az}}\,\exp\left[az-\frac{1}{2az}\left(\nu^2-2ac'-\frac{1}{4}\right)\right]\Leftrightarrow I_{\nu}(az)\,, \tag{47}$$

where $|\lambda|=\sqrt{|\nu|^2+2ac'}$, the short time radial propagator becomes

$$R_{\nu}(r_j,r_{j-1};\epsilon)=\exp\left[\frac{\mathrm{i}M}{2\hbar\epsilon}\left(r_j^2+r_{j-1}^2\right)-\frac{\mathrm{i}M\omega^2\epsilon}{4\hbar}\left(r_j^2+r_{j-1}^2\right)\right]I_{\mu}\left(\frac{Mr_jr_{j-1}}{\mathrm{i}\hbar\epsilon}\right), \tag{48}$$

where

$$\mu = \sqrt{\nu^2 + b} = \sqrt{|m + \lambda - \alpha - \beta' k|^2 + b}\,. \tag{49}$$

In (48) we notice that the exponential factor can be put into the form

$$\exp\left\{ \frac{\mathrm{i}M\omega}{2\hbar} \left(r_j^2 + r_{j-1}^2\right) \frac{1}{\omega\epsilon} \left(1 - \tfrac{1}{2}\,\omega^2\epsilon^2\right) \right\}. \tag{50}$$

Here we let $\eta_j = \dfrac{M\omega r_j}{2\hbar}$ and $\varphi_j = \arcsin(\omega\epsilon)$ and define the υ-function as

$$\upsilon_\mu\left(\eta, \eta'; \varphi\right) = -\mathrm{i}\csc\varphi \exp\Big[\mathrm{i}\left(\eta + \eta'\right)\cot\varphi\Big] I_\mu\left(-2\mathrm{i}\sqrt{\eta\eta'}\,\csc\varphi\right). \tag{51}$$

Then, the short time radial function can be expressed in terms of the υ-function as

$$R_\nu(r_j, r_{j-1}; \epsilon) = \mathrm{e}^{\mathrm{i}\left(\eta + \eta'\right)\cot\varphi} I_\mu\left(-2\mathrm{i}\sqrt{\eta\eta'}\csc\varphi\right) = \mathrm{i}\sin\varphi\,\upsilon_\mu\left(\eta_j, \eta_{j-1}; \varphi\right). \tag{52}$$

Substitution of this into (37) gives

$$Q_\nu\left(r'', r'; \tau\right) = \left(\frac{M\omega}{2\pi\hbar}\right) \lim_{N\to\infty} \int \prod_{j=1}^{N} \upsilon_\mu\left(\eta_j, \eta_{j-1}; \varphi\right) \prod_{j=1}^{N-1} \mathrm{d}\eta_j\,. \tag{53}$$

Since the υ-function satisfies the convolution relation

$$\int_0^\infty \upsilon_\mu\left(\eta'', \eta; \varphi\right) \upsilon_\mu\left(\eta, \eta'; \varphi\right) \mathrm{d}\eta = \upsilon_\mu\left(\eta'', \eta'; 2\varphi\right) \tag{54}$$

the radial integration (53) can be performed, yielding

$$Q_\nu\left(r'', r'; \tau\right) = \frac{M}{2\pi\hbar}\, \upsilon_\mu\left(\eta'', \eta'; \Phi\right), \tag{55}$$

where $\mu = \sqrt{|\nu|^2 + b}$ and $\Phi = \lim_{N\to\infty}(N\varphi) = \arcsin\omega\tau$. In terms of the modified Bessel function,

$$Q_\nu\left(r'', r'; \tau\right) = \frac{M}{2\pi\mathrm{i}\hbar\sin\omega\tau}\, \exp\left[\frac{\mathrm{i}M}{2\hbar}\left(r'^2 + r''^2\right)\cot\omega\tau\right] I_\mu\left(\frac{M\omega r' r''}{\mathrm{i}\hbar\sin\omega\tau}\right). \tag{56}$$

Thus, from (39) we obtain the full propagator with k fixed

$$\begin{aligned} K^{(k)}\left(r'', \chi''; r', \chi'; \tau\right) = {} & \frac{M}{2\pi\mathrm{i}\hbar\sin\omega\tau}\, \exp\left[\frac{\mathrm{i}M}{2\hbar}\left(r'^2 + r''^2\right)\cot\omega\tau\right] \times \\ & \times \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\left(\chi'' - \chi'\right)} I_{|\mu(m)|}\left(\frac{M\omega r' r''}{\mathrm{i}\hbar\sin\omega\tau}\right), \end{aligned} \tag{57}$$

where $\mu(m) = \sqrt{\left|m - \alpha - \beta' k\right|^2 + b}$.

8 Energy Green function

The energy Green function corresponding to the propagator with fixed k is

$$G\left(r'', \chi''; r', \theta\chi'; E, k\right) = \frac{1}{\mathrm{i}\hbar} \int_0^\infty \mathrm{e}^{\mathrm{i}E\tau/\hbar} K^{(k)}\left(r'', \chi''; r', \chi'; \tau\right) \mathrm{d}\tau \tag{58}$$

Upon substitution of (57), this equation may be expressed as

$$G\left(r'', \chi''; r', \chi'; E, k\right) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\left(\chi''-\chi'\right)} G^{(m)}\left(r'', \chi''; r', \chi'; E, k\right), \tag{59}$$

where

$$\begin{aligned} &G^{(m)}\left(r'', \chi''; r', \chi'; E, k\right) = \\ &= -\frac{M}{2\pi\hbar^2} \int_0^\tau \exp\left[\frac{\mathrm{i}M}{2\hbar}\left(r'^2 + r''^2\right)\cot\omega\tau\right] \mathrm{e}^{\mathrm{i}E\tau/\hbar} I_{\mu(m)}\left(\frac{M\omega r' r''}{\mathrm{i}\hbar\sin\omega\tau}\right) \sin\omega\tau\,\mathrm{d}\tau\,. \end{aligned} \tag{60}$$

The τ-integration in the expression for $G^{(m)}$ can be performed by using the integral formula (derived from formula 6.669-4 of Gradshteyn–Ryzhik) [9]

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-\alpha(x+y)\coth q} \mathrm{e}^{-epq} I_{2\nu}\left(2\alpha\sqrt{xy}\,\mathrm{csch}\,q\right) \mathrm{csch}\,q\,\mathrm{d}q = \\ &= \frac{\Gamma\left(p+\nu+\frac{1}{2}\right)}{2\alpha\sqrt{xy}\,\Gamma(2\nu+1)} W_{-p,\nu}(2\alpha x) M_{-p,\nu}(2\alpha y) \end{aligned} \tag{61}$$

valid for $\mathrm{Re}\left(p+\nu+\frac{1}{2}\right) > 0$, $\mathrm{Re}(\nu) > 0$ and $x > y$. Here, $W_{\mu,\nu}(z)$ and $M_{\mu,\nu}(z)$ are the Whittaker functions

$$W_{\mu,\nu}(z) = \frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\nu-\mu\right)} M_{\mu,\nu}(z) + \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)} M_{\mu,-\nu}(z) \tag{62}$$

valid when $\arg|z| < 3\pi/2$ and 2ν is not an integer;

$$W_{-\mu,\nu}(-z) = \frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\nu-\mu\right)} M_{-\mu,\nu}(z) + \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)} M_{\mu,-\nu}(z) \tag{63}$$

valid when $\arg|-z| < 3\pi/2$ and 2ν is not an integer and

$$M_{\mu,\nu}(z) = \mathrm{e}^{-z/2} z^{\nu+1/2} F\left(\tfrac{1}{2}+\nu-\mu, 1+2\nu; z\right), \tag{64}$$

where

$$F(a,b;z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!}\,. \tag{65}$$

$(a)_k$ and $(b)_k$ are the Pochhammer symbols

$$(\alpha)_k \equiv \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha-1)\cdots(\alpha+k-1)\,. \tag{66}$$

Note that the confluent hypergeometric function $F\left(\frac{1}{2}+\nu-\mu, 1+2\nu; z\right)$ vanishes when 2ν is an integer. Letting $2\alpha x = \dfrac{M\omega}{2\hbar}\, r''^2$, $2\alpha y = \dfrac{M\omega}{2\hbar}\, r'^2$, $p = -\dfrac{E}{\hbar\omega}$ and $q = \mathrm{i}\omega\tau$, we get

$$G^{(m)}\left(r'', \chi''; r', \chi'; E, k\right) = \\ = \frac{1}{\pi\hbar\omega}\, \frac{\Gamma\left(p+\frac{1}{2}\mu+\frac{1}{2}\right)}{r'r''\Gamma(\mu+1)}\, W_{-p,\mu/2}\left(\frac{M\omega}{2\hbar}\, r''^2\right) M_{-p,\mu/2}\left(\frac{M\omega}{2\hbar}\, r'^2\right). \tag{67}$$

9 Energy spectrum

The poles of the partial Green function (60), which give the energy spectrum of the system, occur in the Gamma function appearing in the numerator (61). In particular, the Gamma function diverges when

$$\frac{\mu}{2} + \frac{1}{2} - \frac{E}{\hbar\omega} = -n\,, \quad n = 0, 1, 2, \ldots\,. \tag{68}$$

As a result we obtain the energy spectrum

$$E_n = \hbar\omega\left(n + \frac{\mu}{2} + \frac{1}{2}\right) + \frac{(\hbar k)^2}{2M} = \hbar\omega\left(n + \tfrac{1}{2}\sqrt{|(l-\phi-\beta k)/\sigma|^2 + b} + \tfrac{1}{2}\right) + \frac{\hbar^2 k^2}{2M}\,, \tag{69}$$

where $n = 0, 1, 2, \ldots$. Here use was made of $m = l/\sigma$, $\beta' = \beta/\sigma$ and we have taken $q = -\mathrm{e}$ in α leading to $\alpha = -\phi/\sigma$.

10 The wavefunctions

The energy eigenfunctions can be obtained by taking the residues of $G^{(m)}(E,k)$ at the poles except for the phase factor. Substituting condition (64) into (63) the Whittaker function $W_{-p,\mu/2}\left(\dfrac{M}{2\hbar}\, r''^2\right)$ becomes

$$W_{E/\hbar\omega,\mu/2}\left(\frac{M}{2\hbar}\, r''^2\right) = \frac{\Gamma(-\mu)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu+p\right)}\, M_{E/\hbar\omega,\mu/2}\left(\frac{M\omega}{2\hbar}\, r''^2\right) + \\ + \frac{\Gamma(\mu)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+p\right)}\, M_{E/\hbar\omega,-\mu/2}\left(\frac{M\omega}{2\hbar}\, r''^2\right). \tag{70}$$

However, since $\dfrac{\Gamma(\mu)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+p\right)} = 0$ the Whittaker function $W_{E/\hbar\omega,\mu/2}\left(\dfrac{M}{2\hbar}\, r''^2\right)$ reduces to

$$W_{E/\hbar\omega,\mu/2}\left(\frac{M\omega}{2\hbar}\, r''^2\right) = \frac{\Gamma(-\mu)}{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}-\frac{E}{\hbar\omega}\right)}\, M_{E/\hbar\omega,\mu/2}\left(\frac{M\omega}{2\hbar}\, r''^2\right), \tag{71}$$

from which we find

$$G^{(m)}\left(r'',\chi'';r',\chi';E,k\right)=$$

$$=\frac{\Gamma(-\mu)}{\Gamma(\mu+1)}\,\frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{2}-\frac{E}{\hbar\omega}\right)}{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}-\frac{E}{\hbar\omega}\right)}\,\frac{1}{\pi\hbar\omega r'r''}\,M_{E/\hbar\omega,\mu/2}\biggl(\frac{M\omega}{2\hbar}\,r''^2\biggr)M_{E/\hbar\omega,\mu/2}\biggl(\frac{M\omega}{2\hbar}\,r'^2\biggr). \tag{72}$$

The residue of $G^{(m)}\left(r'',\chi'';r',\chi';E,k\right)$ is obtained from

$$\operatorname{Res}\Bigl(G^{(m)}(E)\Bigr)=\lim_{E\to E_n}(E-E_n)G^{(m)}(E)\Big|_{E=E_n}. \tag{73}$$

Taking the limits of the terms in (73), we get

$$\begin{aligned}\lim_{E\to E_n} M_{E/\hbar\omega,\mu/2}\biggl(\frac{M\omega}{2\hbar}\,r'^2\biggr)&=\\ &=\mathrm{e}^{-M\omega r'^2/(4\hbar)}z^{(\mu+1)/2)}\frac{\mu\Gamma(\mu)}{\Gamma(-n)}\sum_{k=0}^{\infty}\frac{\Gamma(k-n)}{\Gamma(k+\mu+1)}\,\frac{1}{k!}\left(\frac{M\omega}{2\hbar}\,r'^2\right)^k=\\ &=M_{-n,\mu+1}\biggl(\frac{M\omega}{2\hbar}\,r'^2\biggr)\end{aligned} \tag{74}$$

and

$$\lim_{E\to E_n}\left[(E-E_n)\,\frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{2}-\frac{E}{\hbar\omega}\right)}{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}-\frac{E}{\hbar\omega}\right)}\right]\Bigg|_{E=E_n}=\frac{(-1)^n 2^{(n-1)/2}\Gamma(1+n+\mu)\sin\pi\mu}{\pi^{3/2}(2n-1)!!}\,, \tag{75}$$

where we have used

$$\Gamma(n+\frac{1}{2})=\frac{\sqrt{\pi}\,(2n-1)!!}{2^{(n-1)/2}}\quad\text{and}\quad\Gamma(-n-\mu)=\frac{1}{\Gamma(1+n+\mu)}\,\frac{\pi}{(-1)^n\sin\pi\mu}\,.$$

With these results, we obtain the residue

$$\begin{aligned}\operatorname{Res}\bigl(G^{(m)}(E)\bigr)&=\frac{\Gamma(-\mu)}{\pi\hbar\omega\Gamma(\mu+1)}\,\frac{(-1)^n 2^{(n-1)/2}\Gamma(1+n+\mu)\sin\pi\mu}{\pi^{3/2}(2n-1)!!}\times\\ &\times\sqrt{\frac{M}{2\pi\hbar^2\omega}}\,M_{-n,\mu+1}\biggl(\frac{M\omega}{2\hbar}\,r'^2\biggr)\sqrt{\frac{M}{2\pi\hbar^2\omega}}\,M_{-n,\mu+1}\biggl(\frac{M\omega}{2\hbar}\,r''^2\biggr).\end{aligned} \tag{76}$$

Finally, the energy eigenfunctions are

$$\begin{aligned}\psi(r,\chi,\zeta)&=C\left[\frac{\Gamma(-\mu)}{\pi\hbar\omega\Gamma(\mu+1)}\,\frac{(-1)^n 2^{(n-1)/2}\Gamma(1+n+\mu)\sin\pi\mu}{\pi^{3/2}(2n-1)!!}\,\frac{M}{2\pi\hbar^2\omega}\right]^{1/2}\times\\ &\times\mathrm{e}^{\mathrm{i}k(\zeta''-\zeta')}\mathrm{e}^{\mathrm{i}m(\chi''-\chi')}M_{-n,\mu+1}\biggl(\frac{M\omega}{2\hbar}\,r^2\biggr),\end{aligned} \tag{77}$$

where $r^2 \equiv r'^2 + r''^2$, $\mu(m) = \sqrt{\left|m - \alpha - \beta' k\right|^2 + b}$, $\zeta'' - \zeta' = z'' - z' + \beta' (\chi'' - \chi)$, $\beta' = \beta/\sigma$, $\alpha = -\phi/\sigma$ (for $q = -\mathrm{e}$), $m = l/\sigma$ and $\chi = \sigma\theta$. In terms of the Confluent Hypergeometric function, the eigenfunctions read

$$\psi_n(r,\theta,z) = A_n \mathrm{e}^{\mathrm{i}k\left(z''-z'\right)} \mathrm{e}^{\mathrm{i}(l+\beta k)\left(\theta''-\theta'\right)}\, r^{\sqrt{\left|(l+\phi-\beta k)/\sigma\right|^2+b}} \exp\left(-\frac{M\omega r^2}{4\hbar}\right) \times$$
$$\times F\left(-n, \sqrt{\left|(l+\phi-\beta k)/\sigma\right|^2 + b} + 1; \frac{M\omega}{2\hbar}\, r^2\right), \tag{78}$$

where

$$A_n = C \left[\frac{\Gamma(-\mu)}{\pi\hbar\omega\Gamma(\mu+1)} \frac{(-1)^n 2^{(n-1)/2}\Gamma(1+n+\mu)\sin\pi\mu}{\pi^{3/2}(2n-1)!!} \left(\frac{M}{2\hbar}\right)^{\mu} \left(\frac{M}{2\pi\hbar^2\omega}\right)\right]^{1/2}$$

and C is a normalization constant.

11 Concluding remarks

In this note path integration was carried out in the field of a topological defect described by a combined dislocation and disclination in solid. The energy Green function was obtained from the propagator of a particle bound in the vicinity of the defect by a short range repulsive and long range attractive force. From the Green function the exact energy spectrum and corresponding eigenfunctions were obtained. The screw dislocation modifies the angular momentum by introducing an additive correction in much the same manner as the magnetic flux. The case of the screw dislocation is very analogous to the well known Aharonov–Bohm (AB) effect and can be interpreted as an extension of the AB effect to include the effects of the gravitational field on quantum systems. The disclination introduces a multiplicative modification $1/\sigma$ to the angular momentum, the appearance of which is understood when we recall that the periodic boundary condition in the space of (5) around the z-axis has periodicity of $2\pi\sigma$ instead of 2π. The presence of both defects cause shifts in the energy spectrum relative to the defect-free Minkowski case even though the particle does not directly touch either of the defects. This last fact further motivates the idea that the topological defects are the gravitational analogy of the AB effect. It is interesting to observe that because of the occurrence of $\beta' (\chi'' - \chi)$ in $(\zeta'' - \zeta')$, the angular wavefunction $\Theta(\theta) \approx \mathrm{e}^{\mathrm{i}(l+\beta k)\left(\theta''-\theta'\right)}$ is modified by $\beta k\,(\theta'' - \theta')$.

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