Nonperturbative approach to (Wiener) functional integral with φ^4 interaction

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We propose the another, in principe nonperturbative, method of the evaluation of the Wiener functional integral for φ^4 term in the action. All infinite summations in the results are proven to be convergent. We find the "generalized" Gelfand–Yaglom differential equation implying the functional integral in the continuum limit.

PACS: 11.15.Tk, 12.38.Lg
Key words: functional integral, φ^4 interaction, nonperturbative approach

We follow the definition of a functional integral as a limit of a sequence of the finite dimensional integrals. This method has no problems with the definition of integral measure, because for the finite dimensional integrals the integral measure is defined correctly. Calculating the finite dimensional integrals we must solve the problem of the calculation of one dimensional integrals

$$I_1 = \int_{-\infty}^{+\infty} \mathrm{d}x \, \exp\{-(ax^4 + bx^2 + cx)\}\,, \tag{1}$$

where $\mathrm{Re}\, a > 0$. The conventional perturbative approach rely on a Taylor's decomposition of x^4 term with consecutive replacements of the integration and summation order. These integrals can be calculated, but the sum is divergent. However, $I_1 = I_1(a,b,c)$ is an entire function *for any* complex values of b and c, since there exist all integrals

$$\partial_c^n \partial_b^m I_1(a,b,c) = (-1)^{n+m} \int_{-\infty}^{+\infty} \mathrm{d}x \, x^{2m+n} \exp\{-(ax^4 + bx^2 + cx)\}\,.$$

Consequently, the power expansions of $I_1 = I_1(a,b,c)$ in c and/or b has an infinite radius of convergence (and in particular they are uniformly convergent on any compact set of values of c and/or b). In what follows we shall use the power expansion in c:

$$I_1 = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \int_{-\infty}^{+\infty} \mathrm{d}x \, x^n \exp\{-(ax^4 + bx^2)\}\,. \tag{2}$$

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The integrals appearing here can be expressed in terms of the parabolic cylinder function $D_\nu(z)$, $\nu = -m - 1/2$, (see, for instance, [2]). For n odd, due to symmetry of the integrand the integrals are zero while for even $n = 2m$ we use:

$$D_{-m-1/2}(z) = \frac{\mathrm{e}^{-z^2/4}}{\Gamma(m+1/2)} \int_0^\infty \mathrm{d}x\, x^{m-1/2} \exp\{-\tfrac{1}{2}x^2 - zx\}\,. \tag{3}$$

Explicitly, for the Eq.(2) we have:

$$I_1 = \frac{\Gamma(1/2)}{\sqrt{b}} \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \mathcal{D}_{-m-1/2}(z)\,, \quad \xi = \frac{c^2}{4b}\,, \quad z = \frac{b}{\sqrt{2a}}\,, \tag{4}$$

we have used the abbreviation:

$$\mathcal{D}_{-m-1/2}(z) = z^{m+1/2}\mathrm{e}^{z^2/4}\, D_{-m-1/2}(z)\,.$$

The sum (4) is convergent for any values of c, b and a positive [1].

Using such expansions we shall calculate the continuum unconditional measure Wiener functional integral:

$$\mathcal{Z} = \int [\mathcal{D}\varphi(x)] \exp(-\mathcal{S})\,,$$

corresponding to the continuum action with the fourth order term:

$$\mathcal{S} = \int_0^\beta \mathrm{d}\tau \left[\frac{c}{2}\left(\frac{\partial\varphi(\tau)}{\partial\tau}\right)^2 + b\varphi(\tau)^2 + a\varphi(\tau)^4\right]. \tag{5}$$

Following the standard procedure, we divide the integration interval into N equal slices. We define the N-dimensional integral by the relation [3]:

$$\mathcal{Z}_N = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \left(\frac{\mathrm{d}\varphi_i}{\sqrt{\frac{2\pi\triangle}{c}}}\right) \exp\left\{-\sum_{i=1}^{N} \triangle \left[\frac{c}{2}\left(\frac{\varphi_i - \varphi_{i-1}}{\triangle}\right)^2 + b\varphi_i^2 + a\varphi_i^4\right]\right\}, \tag{6}$$

where $\triangle = \beta/N$. The continuum Wiener unconditional measure functional integral is defined by the limit:

$$\mathcal{Z} = \lim_{N\to\infty} \mathcal{Z}_N\,.$$

Applying to (6) the relation (3), we obtain, [1]:

$$\mathcal{Z}_N = \left[2\pi(1 + b\triangle^2/c)\right]^{-\frac{N-1}{2}} \left[2\pi(\tfrac{1}{2} + b\triangle^2/c)\right]^{-1/2}\, \mathcal{S}_N \tag{7}$$

with

$$\mathcal{S}_N = \sum_{k_1,\cdots,k_{N-1}=0}^{\infty} \prod_{i=0}^{N} \left[\frac{(\rho)^{2k_i}}{(2k_i)!}\Gamma(k_{i-1} + k_i + \tfrac{1}{2})\mathcal{D}_{-k_{i-1}-k_i-1/2}\,(z)\right], \tag{8}$$

where $k_0 = k_N = 0$, $\rho = (1+b\triangle^2/c)^{-1}$, $z = c(1+b\triangle^2/c)/\sqrt{2a\triangle^3}$ and the argument of the last term $\mathcal{D}_{-k_{N-1}-1/2}(z_{\rm last})$ is $z_{\rm last} = c(\frac{1}{2}+b\triangle^2/c)/\sqrt{2a\triangle^3}$. It is worth while to stress that the factor ρ is independent of the coupling constant. The coupling constant dependence enters into this formula only through the argument z of $\mathcal{D}$ functions.

The question of the convergence of the Eq.(8) is important. Let a_{k_i} is the k_i dependent part of the argument of the product in Eq.(8):

$$a_{k_i} = \frac{(\rho)^{2k_i}}{(2k_i)!}\Gamma(k_{i-1} + k_i + \tfrac{1}{2})\mathcal{D}_{-k_{i-1}-k_i-1/2}\,(z)\,\Gamma(k_{i+1} + k_i + \tfrac{1}{2})\mathcal{D}_{-k_{i+1}-k_i-1/2}\,(z)\,. \tag{9}$$

Using the asymptotic for parabolic cylinder functions and Stirling formula we prove that a_{k_i} approaches to zero in the asymptotic region of k_i as

$$\frac{k_i^D F^{k_i}}{k_i!\ \exp\left(\sqrt{k_i}\right)}\,,$$

where D and F are finite numbers [1]. This asymptotic of a_{k_i} is sufficient for a proof of the uniform convergence of the series not only for single k_i in (8), but for arbitrary tuple $\{k_i\}$ of indices. By the same method we can prove the uniform convergence of the summation over two, three, ..., $N-1$ summation indices k_i in the equation (8). This important conclusion indicates, that in the formula for Z_N the $(N-1)$-tuple summation over k_i's is convergent.

Following the idea of Gelfand and Yaglom the functional integral in the continuum limit is defined by the relation

$$\lim_{N\to\infty} \mathcal{Z}_N = \frac{1}{\sqrt{F(\beta)}}\,,$$

where $F(\beta)$, for our case, is the solution of the differential equation [1]:

$$\frac{\partial^2}{\partial\tau^2}F(\tau) + 4\frac{\partial}{\partial\tau}F(\tau)\,\frac{\partial}{\partial\tau}\ln S(\tau) = F(\tau)\left(\frac{2b}{c} - 2\frac{\partial^2}{\partial\tau^2}\ln S(\tau)\right), \tag{10}$$

calculated at the point β (the upper limit of the time interval in the action (5)). Eq.(10) has to be supplemented by the initial conditions: $F(0) = 1$ and

$$\left.\frac{\partial F(\tau)}{\partial\tau}\right|_{\tau=0} = 0\,.$$

The function $S(\tau)$ is given as the limit $N \to \infty$ of $\mathcal{S}_N$ given by (8).

The result of the summation (8) (replacing the N dimensional integration) is an exact relation, calculated without any approximation. The multiple summations suppress this advantage somewhat, therefore we shall discuss the k_i summations in the formula (8). The details of this calculations are presented in [1].

These summations can be provided by the help of the asymptotic expansions of the parabolic cylinder functions [2]:

$$\mathrm{e}^{z^2/4}\, z^{k_i+1/2}\, D_{-k_i-1/2}(z) = \sum_{j=0}^{\mathcal{J}} (-1)^j \, \frac{(k_i+\frac{1}{2})_{2j}}{j!\,(2z^2)^j}\,, \tag{11}$$

($\mathcal{J}$ is fixed finite number, z is large), and the summation relation for the generating function [2]:

$$\mathrm{e}^{x^2/4} \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!}\, t^k\, D_{-\nu-k}(x) = \mathrm{e}^{(x-t)^2/4}\, D_{-\nu}(x-t)\,. \tag{12}$$

The symbols $(k_i+\frac{1}{2})_{2j}$ and $(\nu)_k$ are the Pochhamer's numbers defined by:

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(k)}\,.$$

In the summation over single k_i in (8), corresponding to the summation over a_{k_i} in (9), there appear a product of two D functions possessing as the function's subscript the summation index. We propose to use the asymptotic expansion for one of them. The asymptotic formula can be used in summation over index k_i provided that the summation runs over finite number of the terms, and the index of the parabolic cylinder function, and its argument obey the relation:

$$k_i < N_0 <\mid z \mid\,, \tag{13}$$

N_0 is an upper bound of summation variables in (8).

This requirement for an application the asymptotic formula is violated if the summation index approaches infinity in (8). We can overcome this problem thank to the uniform convergence of the summations in (8) proven in [1]. We truncate the summation over k_i in (8) but we demand that the partial sum is close to the full infinite one. To minimize the effect of the truncation on the final result in the continuum limit, we have to choose the maximal value of the index k_i such that the remainder of the sum (8) satisfies the inequality

$$\left| \sum_{k_i=N_0+1}^{\infty} a_{k_i} \right| < \varepsilon(N)\,, \tag{14}$$

with $\varepsilon(N)$ approaching zero for large N so that $N^3\varepsilon(N) \to 0$ in the limit $N \to \infty$. We have chosen the third power of N in the inequality in order to eliminate the influence of remainders (14) in Gelfand–Yaglom procedure for the calculation of continuum limit. We estimated the remainder (14) and shown that both conditions (13) and (14) for the application of the proposed summation method are valid simultaneously. In this estimate we utilized the asymptotic form of the parabolic cylinder functions with double asymptotic properties proposed by N. Temme [5].

The result of summations gives for (8) the following relation [1]:

$$\mathcal{Z}_N = \left\{ \prod_{i=0}^{N} \left[2(1 + b\triangle^2/c)\omega_i \right] \right\}^{-1/2} \sum_{\mu=0}^{\mathcal{J}} \frac{(-1)^\mu}{\mu!\,(2z^2)^\mu}\,(N)_{2\mu,0}\;, \tag{15}$$

with the symbols $(N)_{2j,i}$ satisfying the following recurrence relation:

$$(\alpha+1)_{2\mu,p} = \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \omega_\alpha^{-2(\mu-\lambda)} \sum_{i=\max[0,p-2(\mu-\lambda)]}^{2\lambda} \left(\frac{A^2}{\omega_{\alpha-1}\omega_\alpha} \right)^i (\alpha)_{2\lambda,i}\, a_p^{2(\mu-\lambda)+i},$$

$$(\alpha=1)_{2\mu,p} = \frac{a_p^{2\mu}}{\omega_0^{2\mu}}\,, \qquad a_i^j = \binom{j}{i} \frac{(\frac{1}{2})_j}{(\frac{1}{2})_i}\,,$$

where

$$\begin{aligned} \omega_i &= 1 - \frac{A^2}{\omega_{i-1}}\,, \\ A &= \frac{\rho}{2} = \frac{1}{2(1+b\triangle^2/c)}\,, \\ \omega_0 &= 1 - \frac{1}{2(1+b\triangle^2/c)}\,. \end{aligned}$$

$\mathcal{Z}_N$ in (15) is the N-th approximation of the functional integral. The continuum limit of $\mathcal{Z}_N$ we calculate by the procedure proposed by Gelfand–Yaglom following the analogous steps as for the calculation of functional integral for the harmonic oscillator [4]. This means, that for finite N we construct the difference equation for (15), which in the continuum limit $N \to \infty$ is reduced to differential equation. The method and the solutions are discussed in [1], here we present the result only.

The function $S(\tau)$ entering differential equation (10) is calculated by:

$$S(\tau) = \lim_{\triangle\to 0} \sum_{\mu=0}^{\mathcal{J}} \frac{(-1)^\mu}{\mu!\,(2z^2\triangle^3)^\mu} \left(\triangle^{3\mu} (N)_{2\mu,\,0} \right),$$

with the expansion factor

$$\frac{1}{(2z^2\triangle^3)^\mu} = \left(\frac{a\rho}{c^2} \right)^\mu$$

finite in such limit. The additional power $\triangle^{3\mu}$ certifies that in the relation for $(N)_{2\mu,0}$ only the leading terms in powers of $1/\triangle$ survive the continuum limit.

Equation (10) can be simplified by the substitution

$$F(\tau) = \frac{y(\tau)}{S^2(\tau)}\,.$$

For $y(\tau)$ we find the equation

$$\frac{\partial^2}{\partial\tau^2} y(\tau) = y(\tau) \left[\frac{2b}{c} + \left(\frac{\partial}{\partial\tau} \ln S^2(\tau) \right)^2 \right], \tag{16}$$

accompanied by the initial conditions

$$\begin{aligned} y(0) &= S^2(0)\,, \\ \frac{\partial}{\partial\tau}\, y(0) &= \frac{\partial}{\partial\tau}\, S^2(\tau)\Big|_{\tau=0}\,. \end{aligned} \tag{17}$$

The evaluation of function $S(\tau)$ is not yet finished. We present the result, related to the usual perturbative expansion, to the lowest order terms in power expansion in the variable a. Up to the term linear in coupling constant a we have found [1]:

$$\begin{aligned} S^{(1)}(\tau) &= \lim_{\triangle\to 0} \sum_{\mu=0}^{1} \frac{(-1)^\mu}{\mu!\,(2z^2)^\mu}\,(N)_{2\mu,0} = \\ &= 1 - \frac{(\frac{1}{2})_2}{2}\,\frac{a}{4c^2\gamma^3}\left\{\tanh(\tau\gamma) + \tau\gamma\left[3\tanh^2(\tau\gamma) - 1\right]\right\}, \end{aligned}$$

where $\gamma = \sqrt{2b/c}$.

We see, that for $b > 0$ in the harmonic oscillator limit, (i.e. $a = 0$ implying $S^{(1)}(\tau) \equiv 1$), Eqs. (10, 16) reduce to Gelfand–Yaglom equation for the harmonic oscillator. We find that in lowest nontrivial order in a the Eq. (16) is of the modified Bessel type one. Then the functional integral can be expressed in term of the modified Bessel functions.

Inserting the asymptotical perturbative expansion $S^{(1)}(\tau)$ into equation (16), we calculate functional integral beyond simple perturbative expansion. We can render our result as a partial resumation of the simple perturbative series. Moreover, this procedure allows the parameter b take both positive (anharmonic oscillator case) and negative (Higgs case). A similar procedure is familiar in the renormalization group calculations proposed by Gell-Mann and Low. In Callan–Symanzik equation for running coupling constant the β function is calculated perturbatively, however, the equation is solved non-perturbatively.

This work was supported by VEGA projects No. 2/3106/ 2003 (J.B.) and No. 1/025/03 (P.P). The authors acknowledge discussions with J. Polonyi and M. Znojil and their interest in this problem.

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