Feynman path integrals for exponents of polynomially growing functionals

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A general class of functional integrals of the exponents of polynomially growing functionals on the Hilbert space be studied. A representation formula by integrals for Gaussian measures is given for this class of functional integrals. These results are applied to provide a rigorous Feynman path integral representations for the solutions of the time dependent Schrodinger equations with a polynomially growing potentials (it is possible with alternating signs). Special self-adjoint extensions for Schrodinger differential operators with a polynomially growing potentials are obtained.

Let H be a real separable Hilbert space, with inner product $(\cdot,\cdot)$, and norm $\|\cdot\|$, T be a self-adjoint strictly positive trace class operator.

Let $e_1,\, e_2,\, \ldots,\, e_n, \ldots$ be orthonormal base of Hilbert space H such that $Te_n = \lambda_n e_n$, $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n < +\infty$.

Denoted by H_n the span of the first n vectors $e_1,\, e_2,\, \ldots,\, e_n$. Let P_n be the orthogonal projector from the Hilbert space H onto the subspace H_n.

Denoted by μ the Gauss measure on the Hilbert space H with correlation operator T and the zero mean value. The Gauss measure μ is countably additive.

Let $Q_{2l} : H \times \cdots \times H \to R$ be $2l$-linear symmetric continuous function such that $Q_{2l}(x,\ldots,x) > 0$ for all $x \in H$, $x \neq 0$, and $p(x) = (Q_{2l}(x,\ldots,x))^{1/(2l)}$.

Definition. *Let $\upsilon \in \mathbf{C}$, $\upsilon \neq 0$, $\mathrm{Re}\,\upsilon \geq 0$ be a constant and $f : H \to \mathbf{C}$ be a continuous function such that for all natural number n it exists Fresnel integral*

$$\int_{H_n} f(x)\exp\left(-\frac{\upsilon}{2}\,(T^{-1}x,x)\right)\mathrm{d}x = \lim_{\epsilon\to+0}\int_{H_n} f(x)\exp\left(-\frac{\upsilon+\epsilon}{2}\,(T^{-1}x,x)\right)\mathrm{d}x\,.$$

We define the Feynman integral

$$\int_{H} f(x)\exp\left(-\frac{\upsilon}{2}\,(T^{-1}x,x)\right)\mathrm{d}x$$

as the limit

$$I = \lim_{n\to\infty}\frac{\int_{H_n} f(x)\mathrm{e}^{-\upsilon(T^{-1}x,x)/2}\mathrm{d}x}{\int_{H_n}\mathrm{e}^{-\upsilon(T^{-1}x,x)/2}\mathrm{d}x}\,.$$

We put $\int_H f(x)\exp\left(-\frac{1}{2}\,\upsilon(T^{-1}x,x)\right)\mathrm{d}x = I$.

It is easy to see that

$$\int_{H_n}\exp\left(-\frac{\upsilon}{2}\,(T^{-1}x,x)\right)\mathrm{d}x = \sqrt{\left(\frac{2\pi}{\upsilon}\right)^n \lambda_1\ldots\lambda_n}$$

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and

$$\int_H f(x)\exp\left(-\frac{\upsilon}{2}\left(T^{-1}x,x\right)\right)\mathrm{d}x = \lim_{n\to\infty}\frac{\int_{H_n} f(x)\exp\left(-\frac{\upsilon}{2}\left(T^{-1}x,x\right)\right)\mathrm{d}x}{\sqrt{\left(\frac{2\pi}{\upsilon}\right)^n\lambda_1\ldots\lambda_n}}\,.$$

Let $F(H)$ be the space of all continuous functions $f: H\to\mathbf{C}$ such that for all $h_1,h_2\in H$ the function $\varphi(\alpha,\beta)=f(\alpha h_1+\beta h_2)$ have analytic continuation $\mathbf{C}\times\mathbf{C}$ and for all $\alpha,\beta\in\mathbf{C}$. It is true

$$|\varphi(\alpha,\beta)|\le c_1\exp\Big(c_2\left[|\alpha|\,\|h_1\|\right]^{2l-\varepsilon}+c_3\left[|\beta|\|h_2\|\right]^{2l-\varepsilon}\Big)\,,$$

where constant $c_1>0$, $c_2>0$, $c_3>0$, $0<\varepsilon<1$ depend only of the function f.

Lemma 1. *It exists a sequence of positive numbers $\alpha_n\le\lambda_n$ such that we have*

$$\int_H\left[\prod_{n=1}^{\infty}\left(1+\alpha_n t\frac{\|x\|^2}{p(Sx)^2}\right)\right]^2\exp\left(-\frac{\|x\|^4}{p(Sx)^2}\right)\mu(\mathrm{d}x)\le\mathrm{e}^t$$

for all real $t\ge0$ and for all self-adjoint strictly positive bounded operator $S: H\to H$.

Proof. Let us take $a_0=1$,

$$a_n=n^{-2n}\left[1+\sum_{k=1}^{n}\int_H\frac{1}{\|x\|^{2k}}\,\mu(\mathrm{d}x)\right]^{-1},$$

$\alpha_n=2^{-2n}a_{2n}$ for all natural number n. We have

$$\begin{aligned}\left[\prod_{n=1}^{\infty}\left(1+\alpha_n t\,\frac{\|x\|^2}{p(Sx)^2}\right)\right]^2&\exp\left(-\frac{\|x\|^4}{p(Sx)^2}\right)\le\\ &\le\sum_{n=0}^{\infty}a_n\left(t\frac{\|x\|^2}{p(Sx)^2}\right)^n\exp\left(-\frac{\|x\|^4}{p(Sx)^2}\right)\le\\ &\le\sum_{n=0}^{\infty}a_nt^n\frac{1}{\|x\|^{2n}}\left[\frac{\|x\|^4}{p(Sx)^2}\right)^n\exp\left(-\frac{\|x\|^4}{p(Sx)^2}\right)\le\sum_{n=0}^{\infty}n^na_nt^n\frac{1}{\|x\|^{2n}}\,.\end{aligned}$$

Hence

$$\begin{aligned}\int_H\left[\prod_{n=1}^{\infty}\left(1+\alpha_n t\frac{\|x\|^2}{p(Sx)^2}\right)\right]^2&\exp\left(-\frac{\|x\|^4}{p(Sx)^2}\right)\mu(\mathrm{d}x)\le\\ &\le\sum_{n=0}^{\infty}n^na_nt^n\int_H\frac{1}{\|x\|^{2n}}\mu(\mathrm{d}x)\le\sum_{n=0}^{\infty}n^{-n}t^n\le\mathrm{e}^t\,.\end{aligned}$$

Lemma 2. *Let d be a natural number. Then it exists a self-adjoint strictly positive bounded operator S such that*

$$Se_n=\gamma_ne_n\,,\qquad\alpha_n\ge\gamma_n\ge\gamma_{n+1}>0\,,$$

$$\frac{\|P_nx\|^2}{(p(SP_nx))^2}\le2\frac{\|x\|^2}{(p(Sx))^2}$$

and

$$\int_H \exp\left(\left[\frac{\|T^{-1}Sx\|}{p(Sx)}\right]^d\right)\mu(\mathrm{d}x) \le +\infty\,.$$

Proof. Let $\gamma_1 = \alpha_1$, and $I = \exp\left(\left[\frac{\lambda_1^{-1}}{p(e_1)}\right]^d\right)$. It is easy to see that

$$I = \frac{1}{\sqrt{2\pi\lambda_1}}\int_{-\infty}^{+\infty}\exp\left(\left[\frac{\lambda_1^{-1}\gamma_1|x_1|}{p(\gamma_1 x_1 e_1)}\right]^d\right)\exp\left(-\frac{x_1^2}{2\lambda_1}\right)\mathrm{d}x_1\,.$$

Suppose that we have $\gamma_1, \ldots, \gamma_n$ such that $\alpha_1 \ge \gamma_1 > 0, \ldots, \alpha_n \ge \gamma_n > 0$, $\gamma_1 \ge \gamma_2 \ge \ldots \ge \gamma_n > 0$ and

$$\begin{aligned} I_k = {} & \frac{1}{\sqrt{(2\pi)^n\lambda_1\ldots\lambda_{k-1}\lambda_k}}\times \\ & \times\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\exp\left(\left[\frac{\sqrt{(\lambda_1^{-1}\gamma_1 x_1)^2+\ldots+(\lambda_k^{-1}\gamma_k x_k)^2}}{p(\gamma_1 x_1 e_1+\ldots+\gamma_k x_k e_k)}\right]^d\right)-\right. \\ & \left.-\exp\left(\left[\frac{\sqrt{(\lambda_1^{-1}\gamma_1 x_1)^2+\ldots+(\lambda_{k-1}^{-1}\gamma_{k-1}x_{k-1})^2}}{p(\gamma_1 x_1 e_1+\ldots+\gamma_{k-1}x_{k-1}e_{k-1})}\right]^d\right)\right|\times \\ & \times\exp\left(-\frac{x_1^2}{2\lambda_1}-\ldots-\frac{x_{k-1}^2}{2\lambda_{k-1}}-\frac{x_k^2}{2\lambda_k}\right)\mathrm{d}x_1\ldots\mathrm{d}x_{k-1}\mathrm{d}x_k < 2^{-k} \end{aligned}$$

for all $k = 2, 3, \ldots, n$.

Let us find γ_{n+1}. Denoted by

$$\begin{aligned} I_{n+1}(t) = {} & \frac{1}{\sqrt{(2\pi)^{n+1}\lambda_1\ldots\lambda_n\lambda_{n+1}}}\times \\ & \times\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\exp\left(\left[\frac{\sqrt{(\lambda_1^{-1}\gamma_1 x_1)^2+\ldots++(\lambda_{n+1}^{-1}t x_{n+1})^2}}{p(\gamma_1 x_1 e_1+\ldots+\gamma_n x_n e_n+t x_{n+1}e_{n+1})}\right]^d\right)-\right. \\ & \left.-\exp\left(\left[\frac{\sqrt{(\lambda_1^{-1}\gamma_1 x_1)^2+\ldots+(\lambda_n^{-1}\gamma_n x_n)^2}}{p(\gamma_1 x_1 e_1+\ldots+\gamma_n x_n e_n)}\right]^d\right)\right|\times \\ & \times\exp\left(-\frac{x_1^2}{2\lambda_1}-\ldots-\frac{x_n^2}{2\lambda_n}-\frac{x_{n+1}^2}{2\lambda_{n+1}}\right)\mathrm{d}x_1\ldots\mathrm{d}x_n\mathrm{d}x_{n+1} \end{aligned}$$

for all $t \ge 0$.

By Lebesgue's dominated convergence theorem we get

$$\lim_{t\to+0} I_{n+1}(t) = 0\,.$$

Thus it exist $t > 0$ such that $I_{n+1}(t) < 2^{-(n+1)}$, $t \le \min(a_{n+1}, \gamma_n)$ and

$$\frac{\|P_n x\|^2}{(p(SP_n x))^2} \le \frac{2 - \frac{1}{n+1}}{2 - \frac{1}{n}} \, \frac{\|P_{n+1} x\|^2}{(p(SP_{n+1} x))^2} \, .$$

We can take $\gamma_{n+1} = t$. Thus we find $\gamma_1, \ldots, \gamma_n, \ldots$.

Let us take $d = 6$. By lemma 2 we find self-adjoint strictly positive bounded operator S such that $Se_n = \gamma_n e_n$. Let $t = t_1 + t_2 \mathrm{i} \in \mathbf{C}$, $t_1, t_2 \in \mathbf{R}$, $t_1 \ge 0$, $t \ne 0$ and $x = \sum_{n=1}^{\infty} x_n e_n \in H$, $x \ne 0$.

We suppose

$$\begin{aligned} a(x) &= \left(1 + t_1 \frac{\|x\|^2}{p(Sx)^2}\right)^2 + \left(t_2 \frac{\|x\|^2}{p(Sx)^2}\right)^2 , \\ A_1(x) &= \sum_{n=1}^{\infty} \frac{1 + t_1 \frac{\|x\|^2}{p(Sx)^2}}{a(x)} \, \gamma_n x_n e_n \end{aligned}$$

and

$$A_2(x) = \sum_{n=1}^{\infty} \frac{t_2 \frac{\|x\|^2}{p(Sx)^2}}{a(x)} \, \gamma_n x_n e_n \, .$$

We take

$$J_1(t, x) = \prod_{n=1}^{\infty} \left(1 + \gamma_n t \frac{\|x\|^2}{p(Sx)^2}\right)$$

and

$$\begin{aligned} J_2(t, x) &= 1 + 2t \frac{(x, A_1(x)) + \mathrm{i}(x, A_2(x))}{p(Sx)^2} - \\ &\quad - 2t \frac{\|x\|^2}{p(Sx)^{2l+2}} \Big[Q(SA_1(x), Sx, \ldots, Sx) + \mathrm{i} Q(SA_2(x), Sx, \ldots, Sx) \Big]. \end{aligned}$$

Theorem 1. *If a function f belongs to the space $F(H)$, then for all $\nu, \upsilon \in \mathbf{C}$, $\nu \ne 0$, upsilon $\ne 0$,$\mathrm{Re}\, \nu \ge 0$, $\mathrm{Re}\, \upsilon \ge 0$ the Feynman integral*

$$\int_H f(x) \exp\left(-\nu (p(x))^{2l}\right) \exp\left(-\frac{\upsilon}{2} \, (T^{-1} x, x)\right) \mathrm{d}x$$

exists and it can also be represented by

$$\begin{aligned} \int_H f(x) \exp\left(-\nu (p(x))^{2l}\right) \exp\left(-\frac{\upsilon}{2} \, (T^{-1} x, x)\right) \mathrm{d}x &= \int_H f\left(\sigma x + \omega \, \frac{\|x\|^2}{(p(Sx))^2} \, Sx\right) \times \\ \times \exp\left(-\nu \left(p(\sigma x + \omega \, \frac{\|x\|^2}{(p(Sx))^2} Sx)\right)^{2l}\right) &\exp\left(-\frac{\omega}{\sigma} \, \frac{\|x\|^2}{(p(Sx))^2} (x, T^{-1} Sx)\right) \times \\ \times \exp\left(-\frac{\upsilon \omega^2}{2} \, \frac{\|x\|^4}{(p(Sx))^4} \, (Sx, T^{-1} Sx)\right) &J_1\left(\frac{\omega}{\sigma}, x\right) J_2\left(\frac{\omega}{\sigma}, x\right) \mu(\mathrm{d}x) \, , \end{aligned}$$

where $\sigma = \upsilon^{-1/2}$, $\omega = 2^{4l} r \nu^{-1/(2l)}$, $r \in \mathbf{R}$, $r > |\nu| + |\upsilon| + 1$.

Proof. Let $\nu, \upsilon \in \mathbf{R}$, $\nu > 0$, $\upsilon > 0$, $r \in \mathbf{R}$, $r > |\nu| + |\upsilon| + 1$ and $\sigma = \upsilon^{-1/2}$, $\omega = 2^{4l} r \nu^{-1/(2l)}$.

Let $x' = \sigma x + \omega \, \dfrac{\|x\|^2}{(p(Sx))^2} \, Sx$ for all $x \in H_n$. We have

$$\frac{1}{\sqrt{(\frac{2\pi}{\upsilon})^n \, \lambda_1 \dots \lambda_n}} \int_{H_n} f(x') \exp\left(-\nu (p(x'))^{2l}\right) \exp\left(-\frac{\upsilon}{2}(T^{-1}x', x')\right) \mathrm{d}x' =$$

$$= \frac{1}{\sqrt{(2\pi)^n \lambda_1 \dots \lambda_n}} \int_{H_n} f\left(\sigma x + \omega \frac{\|x\|^2}{(p(Sx))^2} Sx\right) \exp\left(-\nu \left(p(\sigma x + \omega \frac{\|x\|^2}{(p(Sx))^2} Sx)\right)^{2l}\right)$$

$$\times \exp\left(-\frac{\omega}{\sigma} \, \frac{\|x\|^2}{(p(Sx))^2} (x, T^{-1}Sx) - \frac{\upsilon \omega^2}{2} \, \frac{\|x\|^4}{(p(Sx))^4} \, (Sx, T^{-1}Sx)\right) \times$$

$$\times J_1\left(\frac{\omega}{\sigma}, x\right) J_2\left(\frac{\omega}{\sigma}, x\right) e^{-(T^{-1}x, x)/2} \mathrm{d}x \, .$$

Taking the limit for $n \to \infty$ we prove the theorem 1. The convergence follow from the lemma 1 and the lemma 2.

We say that a function

$$v : [0, +\infty) \times \mathbf{R}^n \to \mathbf{R}$$

satisfies condition (A_k) if

$$v(t, q) = v_1(t, q) + v_2(t, q) \, ,$$

where for all $t \geq 0$ the function $v_1(t, .)$ is nondegenerate homogeneous polynomial of degree $2k$ for argument q and the function $v_2(t, \cdot)$ is polynomial of degree less then $2k$.

Consider Schrödinger equation

$$\frac{\partial u(t, q)}{\partial t} = \tfrac{1}{2} \, \mathrm{i} \Delta u(t, q) - \mathrm{i} v(t, q) u(t, q) \qquad (1)$$

with a potential

$$v : [0, +\infty) \times \mathbf{R}^n \to \mathbf{R} \, .$$

Denote by E_t the Banach space of all continuous mapping $x : [0, t] \in \mathbf{R}^n$ such that $x(0) = x(t) = 0$.

Theorem 2. *If the potential* v *is continuous and satisfy condition* (A_k) *for a naturel* k *then it exists for the Schrödinger equation (1) the Green's function*

$$U(t, q_0, q) = \int_{E_t} \exp\left[-\mathrm{i} \int_0^t v\left(\tau, x(\tau) + q_0 + \frac{\tau}{t} \, (q - q_0)\right) \mathrm{d}\tau\right] \times$$

$$\times \exp\left[-\frac{\mathrm{i} \|q - q_0\|^2_{\mathbf{R}^n}}{2t} + \frac{\mathrm{i}}{t} \int_0^t \left(q - q_0, x(\tau)\right)_{\mathbf{R}^n} \mathrm{d}\tau\right] \exp\left[-\frac{\mathrm{i}}{2} \int_0^t \|x'(\tau)\|^2_{\mathbf{R}^n} \mathrm{d}\tau\right] \mathrm{d}x$$

and

$$U\,(0,q_0,q)=\delta(q-q_0)\,.$$

Remark. *The Green's function $U(t,q_0,q)$ defines special self-adjoint extension for the Schrödinger differential operator $\triangle+v$ in the case that this operator is not essentially self-adjoint.*

Consider on the Hilbert space $Q=H$ the Cauchy problem

$$\frac{\partial u(t,q)}{\partial t}=\frac{\mathrm{i}}{2}\,\Delta_T u(t,q)-\mathrm{i}V(t,q)u(t,q) \tag{2}$$

of Schrödinger equation with a potential $V:[0,+\infty)\times Q\to\mathbf{R}$ and with a initial condition

$$u(0,q)=u_0(q-q_0)\quad(\forall\,q\in Q)\;, \tag{3}$$

where $u_0:Q\to\mathbf{C}$,

$$\Delta_T=\sum_{n=1}^{\infty}\lambda_n\frac{\partial^2}{\partial q_n^2}\qquad q=\sum_{n=1}^{\infty}q_ne_n\in Q\,.$$

Denote by F_t the Banach space of all continuous mapping $\xi:[0,t]\in Q$ such that $\xi(t)=0$.

Let $V(t,q)=V_1(t,q,\ldots,q)+V_2(t,q)$, where

$$V_1:[0,+\infty)\times Q\times\cdots\times Q\in\mathbf{R}\,,$$

$V_2:[0,+\infty)\times Q\to\mathbf{R}$ are continuous functions and for all $t\geq 0$ the function $V_1(t,.,\ldots,.)$ is $2k$-linear symmetric nondegenerate and the function $V_1(t,.)$ is polynomial of degree less then $2k$ for argument q.

Theorem 3. *If a function u_0 belongs to the space $F(Q)$ then it exists a solution of the Cauchy problem of the Schrödinger equation (2), (3) and*

$$u(t,q)=\int_{F_t}u_0\big(\xi(0)+q\big)\exp\!\left(-\mathrm{i}\int_0^t V\big(\tau,\xi(\tau)+q\big)\mathrm{d}\tau\right)\times$$
$$\times\exp\!\left(-\mathrm{i}\int_0^t\left\|T^{-1/2}\xi'(\tau)\right\|_Q^2\mathrm{d}\tau\right)\mathrm{d}\xi\,.$$

References

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